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A Criterion for Reducibility of Matrices

Vladimir Monov[1]*, *Michael Tsatsomeros*[2]

[1] *Institute of Information Technologies, 1113 Sofia*
E-mail: vmonov@iit.bas.bg
[2] *Mathematics Department, Washington State University, Pullman, WA 99164-3113, USA*
E-mail: tsat@math.wsu.edu

Abstract: *The problem of existence and characterization of non-trivial reducing subspaces for a given matrix is studied employing some basic tools of multilinear algebra. A criterion for reducibility of a single matrix is obtained which is also extended to the case of simultaneous reduction of two or more matrices.*

Keywords: *reducing subspace, invariant subspace, compound matrix, Grassmann space, decomposable vector.*

Introduction

The notions of invariant and reducing subspaces are well known in linear algebra and matrix theory. Invariant and reducing subspaces play a key role in studying the spectral properties and canonical forms of matrices and have a number of important applications [2]. The problem of existence of a non-trivial subspace which is a common invariant subspace of two or more matrices is of considerable interest and is treated in [1] and [6]. Under certain assumptions, a procedure to check whether such a subspace exists is proposed in [1], and a general necessary and sufficient condition is obtained in [6]. The approach developed in the latter reference is based on the concepts of multilinear algebra and utilizes certain properties of Grassmann representatives of the invariant subspace. In the present paper, this approach is further developed to characterize and study the existence of non-trivial reducing subspaces. In particular, we have obtained a criterion for reducibility of a single matrix which is also applicable to the case of simultaneous reduction of several matrices. The next section of the paper contains the necessary background theory and the main result is stated in Section 3. The application of the criterion is illustrated by an example in Section 4.

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2. Preliminaries

Let $\langle ... \rangle$ be the usual inner product on $\boldsymbol{C}^n$, i.e., $\langle x, y\rangle = x^*y$, where * denotes complex conjugate transposition. Recall first that the *sum* of two subspaces L and M of $\boldsymbol{C}^n$ is defined as $L + M = \{z \in \boldsymbol{C}^n : z = x + y, x \in L, y \in M\}$. The sum is said to be *direct* if $L \cap M = \{0\}$, in which case it is denoted by $L \dot{+} M$. The subspaces L and M are complementary (direct complements) if $L \cap M = \{0\}$ and $L \dot{+} M = \boldsymbol{C}^n$. Subspaces L and M are *orthogonal* if $\langle x, y\rangle = 0$ for every $x \in L$ and $y \in M$; they are *orthogonal complements* if, in addition, they are complementary. In the latter case we write $L = M^{\perp}$ and $M = L^{\perp}$.

For any $A \in \boldsymbol{C}^{n\times n}$ and $S \subseteq \boldsymbol{C}^n$, AS denotes the set $\{Ax : x \in S\}$. A subspace $L \subseteq \boldsymbol{C}^n$ is *invariant* for $A \in \boldsymbol{C}^{n\times n}$ (or *A-invariant*}) if $AL \subseteq L$. An *A-invariant* subspace L is *A-reducing* if there exists a direct complement M to L in $\boldsymbol{C}^n$ that is also *A-invariant*; the pair of subspaces (L, M) is then called a *reducing pair* for A. Clearly, $\{0\}$, $\boldsymbol{C}^n$ and the generalized eigenspaces of $A \in \boldsymbol{C}^{n\times n}$ are examples of A-reducing subspaces.

The following basic notation and facts from multilinear algebra will be used; see e.g., [4]. Given positive integers $k \le n$, let $Q_{k,n}$ be the set of all k-tuples of $\{1,...,n\}$ with elements in increasing order. The members of $Q_{k,n}$ are considered ordered lexicographically.

For any matrix $X \in \boldsymbol{C}^{n\times n}$ and nonempty $\alpha \subseteq \{1,..., m\}$, $\beta \subseteq \{1,..., n\}$, let $X[\alpha|\beta]$ denote the submatrix of X in rows and columns indexed by α and β respectively. Given an integer $0 < k \le \min\{m, n\}$, the k-th *compound* of X is defined as the $\binom{m}{k}\times\binom{n}{k}$ matrix

$$X^{(k)} = (\det X[\alpha|\beta])_{\alpha\in Q_{k,m},\beta\in Q_{k,n}}.$$

Matrix compounds satisfy $(XY)^{(k)} = X^{(k)}Y^{(k)}$. The *exterior product* of the vectors $x_i \in \boldsymbol{C}^n$, $i = 1,..., k$, denoted by $x_1\wedge ...\wedge x_k$ is the $\binom{n}{k}$-component vector equal to the k-th compound of $X = [x_1 |...| x_k]$; i.e.,

$$x_1\wedge ...\wedge x_k = X^{(k)}.$$

Consequently, if $A \in \boldsymbol{C}^{n\times n}$ and $0 < k \le n$, the first column of $A^{(k)}$ is precisely the exterior product of the first k columns of A. Exterior products satisfy the following:

(1) $$x_1\wedge ...\wedge x_k = 0 \leftrightarrow x_1,..., x_k \text{ are linearly dependent.}$$

(2) $$\mu_1 x_1\wedge ...\wedge \mu_k x_k = \prod_{i=1}^{k} \mu_i\, (x_1\wedge ...\wedge x_k),\ \mu_i \in \boldsymbol{C}.$$

(3) $$A^{(k)}(x_1\wedge ...\wedge x_k) = Ax_1\wedge ...\wedge Ax_k .$$

When a vector $x \in \boldsymbol{C}^{\binom{n}{k}}$ is viewed as a member of the k-th Grassmann space over $\boldsymbol{C}^n$, it is called *decomposable* if $x = x_1\wedge ...\wedge x_k$ for some $x_i \in \boldsymbol{C}^n$, $i = 1, ..., k$. We refer to $x_1,..., x_k$ as the *factors* of x. By conditions (2) and (3), those decomposable vectors whose factors are linearly independent eigenvectors of $A \in \boldsymbol{C}^{n\times n}$ are eigenvectors of $A^{(k)}$. The spectrum of $A^{(k)}$ coincides with the set of all possible k-products of the eigenvalues of A. In general, not all eigenvectors of a matrix compound are decomposable.

Consider now a k-dimensional subspace $L \subseteq \boldsymbol{C}^n$ spanned by $\{x_1,..., x_k\}$. By (1) and the definition of the exterior product it follows that

$$\mathcal{L} = \{ x \in \boldsymbol{C}^n : x \wedge x_1 \wedge ... \wedge x_k = 0 \}.$$

The vector $x_1 \wedge ... \wedge x_k$ is known as a *Grassmann representative of* $\mathcal{L}$. As a consequence, two k-dimensional subspaces spanned by $\{x_1,..., x_k\}$ and $\{y_1,..., y_k\}$ respectively, coincide if and only if for some nonzero $\mu \in \boldsymbol{C}$,

$$x_1 \wedge ... \wedge x_k = \mu \, (y_1 \wedge ... \wedge y_k);$$

that is, Grassmann representatives for a subspace differ only by a nonzero scalar factor.

Finally, let $A \in \boldsymbol{C}^{n \times n}$ and let $\mathcal{L} \subseteq \boldsymbol{C}^n$ be an A-invariant subspace with basis $\{x_1,..., x_k\}$. We shall use the fact that any Grassmann representative of $\mathcal{L}$ is an eigenvector of A^k. This is seen by noting that if $A\mathcal{L} \subseteq \mathcal{L}$, then properties (1) and (3) imply that $A^{(k)} (x_1 \wedge ... \wedge x_k)$ is either 0 or a Grassmann representative of $\mathcal{L}$; that is, $A^{(k)} (x_1 \wedge ... \wedge x_k)$ is indeed a scalar multiple of $x_1 \wedge ... \wedge x_k \neq 0$.

3. Reducing subspaces

In this section, we present reducibility conditions for a matrix and for a pair of matrices based on a relationship between Grassmann representatives of reducing subspaces and eigenvectors of matrix compounds. First is an auxiliary result characterizing complementary subspaces.

Lemma 1. Let $\mathcal{L}$, $\mathcal{M} \in \boldsymbol{C}^n$ be subspaces with $\dim \mathcal{L} = k$ and $\dim \mathcal{M} = n - k$, $1 \leq k < n$, and let $x, y \in \boldsymbol{C}^{\binom{n}{k}}$ be Grassmann representatives of $\mathcal{L}$ and $\mathcal{M}^\perp$, respectively. The following are equivalent.

(i) $\mathcal{L}$ and $\mathcal{M}$ are direct complements in $\boldsymbol{C}^n$;

(ii) vectors x and y satisfy $\langle x, y \rangle \neq 0$.

P r o o f. Since $\dim \mathcal{L} + \dim \mathcal{M} = n$, condition (i) is equivalent to

$$\mathcal{L} \cap \mathcal{M} = \{0\}.$$

Let $\{x_1,..., x_k\}$ and $\{y_1,..., y_k\}$ be bases of $\mathcal{L}$ and $\mathcal{M}^\perp$, respectively, and consider the $n \times k$ matrices $X = [x_1|...|x_k]$ and $Y = [y_1|...|y_k]$. Then, up to nonzero scalar multiples, $x = x_1 \wedge ... \wedge x_k$ and $y = y_1 \wedge ... \wedge y_k$. By Cauchy-Binet's formula for the determinant it can be seen that $\langle x, y \rangle = \det X^*Y$. Hence, in order to prove the lemma, we need only show that (5) is equivalent to $\det X^*Y \neq 0$.

Assume first that X^*Y is singular. Then there exists a nonzero vector $u \in \boldsymbol{C}^k$ such that $u^*X^*Y v = \langle Xu, Yv \rangle = 0$ for all $v \in \boldsymbol{C}^k$.Thus, $0 \neq Xu \in \mathcal{L}$ and since $\mathcal{M}^\perp = \{Y v : v \in \boldsymbol{C}^k\}$, it follows that $Xu \in \mathcal{M}$, which contradicts to (5). Conversely, if $\mathcal{L}$ and $\mathcal{M}$ have a common nonzero vector z, then $z = Xu$ for some $u \in \boldsymbol{C}^k$ and also z is orthogonal to all vectors in $\mathcal{M}^\perp$; i.e., $u^*X^*Y v = 0$ for all $v \in \boldsymbol{C}^k$. This implies that X^*Y is singular.

Recall that if $A \in \boldsymbol{C}^{n \times n}$ and λ, μ are distinct eigenvalues of A, then by the biorthogonality principle (see e.g., [3], each left eigenvector of A corresponding to μ is orthogonal to each right eigenvector of A corresponding to λ.

Theorem 1. Let $A\in \boldsymbol{C}^{n\times n}$ and $1\leq k < n$. The following are equivalent:

(i) there exist subspaces $\mathcal{L}$, $\mathcal{M}\subset \boldsymbol{C}^{n}$, of dimensions k and $n-k$, respectively, such that $(\mathcal{L}, \mathcal{M})$ is a reducing pair for A;

(ii) for all $s\in \boldsymbol{C}^{k}$, $(A+sI)^{(k)}$ has a pair of right and left eigenvectors $x, y\in \boldsymbol{C}^{\binom{n}{k}}$ that are decomposable and satisfy $\langle x, y\rangle \neq 0$;

(iii)} there exists $\hat{s}\in \boldsymbol{C}$ such that $(A+\hat{s}I)$ is nonsingular and $(A+\hat{s}I)^{(k)}$ has a pair of right and left eigenvectors $x, y\in \boldsymbol{C}^{\binom{n}{k}}$ that are decomposable and satisfy $\langle x, y\rangle \neq 0$.

Moreover, when either of these conditions hold, x and y in (ii)} and (iii) are Grassmann representatives of $\mathcal{L}$ and $\mathcal{M}^{\perp}$, respectively, and they correspond to the same eigenvalue of $(A+sI)^{(k)}$.

Proof. (i) $\Rightarrow$(ii). Let $(\mathcal{L}, \mathcal{M})$ be a reducing pair for A with $\dim \mathcal{L}=k$ and $\dim \mathcal{M}=n-k$; i.e., $\mathcal{L}$ and $\mathcal{M}$ are A-invariant subspaces that are complementary in $\boldsymbol{C}^n$. Let $\{x_1,..., x_k\}$ and $\{y_1,..., y_k\}$ be bases of $\mathcal{L}$ and $\mathcal{M}^{\perp}$, respectively. Since $A\mathcal{L}\subseteq \mathcal{L}$ if and only if $(A+sI)\mathcal{L}\subseteq \mathcal{L}$ for all $s\in \boldsymbol{C}$, it follows by the discussion in Section 2, that $x=x_1\wedge...\wedge x_k$ is a right eigenvector of $(A+sI)^{(k)}$. Similarly, $(A+sI)\mathcal{M}\subseteq \mathcal{M}$ for all $s\in \boldsymbol{C}$, which is also equivalent to $(A+sI)^*\mathcal{M}^{\perp}\subseteq \mathcal{M}^{\perp}$. Thus, $y=y_1\wedge...\wedge y_k$ is a right eigenvector of $((A+sI)^*)^{(k)}$; due to the compound matrix property $((A+sI)^*)^{(k)} = ((A+sI)^{(k)})^*$, we have that y is a left eigenvector of $(A+sI)^{(k)}$. Since $\mathcal{L}$ and $\mathcal{M}$ are complementary, it follows by Lemma , that $\langle x, y\rangle \neq 0$.

(ii) $\Rightarrow$(iii). Follows trivially.

(i) $\Rightarrow$(ii). Let $\hat{s}$ be such that $(A+\hat{s}I)$ is nonsingular and let $x=x_1\wedge...\wedge x_k$, $y=y_1\wedge...\wedge y_k$ be right and left eigenvectors of $(A+\hat{s}I)^{(k)}$, respectively, such that $\langle x, y\rangle \neq 0$.

Then $(A+\hat{s}I)^{(k)}$ is nonsingular and there exists nonzero $\lambda\in \boldsymbol{C}$ such that

(6) $$(A+\hat{s}I)^{(k)}x = (A+\hat{s}I)x_1\wedge...\wedge (A+\hat{s}I)x_k = \lambda (x_1\wedge...\wedge x_k).$$

By the biorthogonality principle, $\langle x, y\rangle \neq 0$ implies that y corresponds to the same eigenvalue λ of $(A+\hat{s}I)^{(k)}$, i.e.,

(7) $$\begin{aligned}((A+\hat{s}I)^{(k)})^* y &= ((A+\hat{s}I)^*)^{(k)} y\\ &= (A+\hat{s}I)^* y_1\wedge...\wedge (A+\hat{s}I)^* y_k\\ &= \overline{\lambda} (y_1\wedge...\wedge y_k).\end{aligned}$$

By (6) it follows that the subspace spanned by $\{x_1,..., x_k\}$ coincides with the subspace spanned by $\{(A+\hat{s}I)x_1,..., ((A+\hat{s}I)x_k\}$. Thus, $\mathcal{L}=\operatorname{span}\{x_1,..., x_k\}$ is an invariant subspace of $(A+\hat{s}I)$ and hence of A. Similarly, it follows from (7) that $\operatorname{span}\{y_1,..., y_k\}$ is an invariant subspace of A^* or, equivalently, that $\mathcal{M}=(\operatorname{span}\{y_1,..., y_k\})^{\perp}$ is an invariant subspace of A. By Lemma 1, the inequality $\langle x, y\rangle \neq 0$ implies that the A-invariant subspaces $\mathcal{L}$ and $\mathcal{M}$ are direct complements, completing the proof.

Given an integer k, $1\leq k < n$, the classical Schur triangularization theorem shows that every matrix $A\in \boldsymbol{C}^{n\times n}$ has an A-invariant subspace of dimension k. However, considering the Jordan normal form of a matrix, it can be seen that not every $A\in\mathbf{C}^{n\times n}$ has an A-reducing subspace of arbitrary dimension k. For instance, if A is an n-dimensional Jordan block, it can be shown that the only A-reducing subspaces are $\{0\}$ and $\mathbf{C}^n$. In the next example, we illustrate how Theorem 1 can be employed to rule out the existence of reducing subspaces of a certain dimension.

Example 1. Let n = 4, k = 1 and

$$A=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The distinct eigenvalues of A are $\lambda_1=1$ and $\lambda_2=-1$; $A^{(1)}=A$ has a pair of right and left eigenvectors $x_1=[1\ 0\ 0\ 0]^T$ and $y_1=[0\ 1\ 0\ 0]^T$ corresponding to $\lambda_1=1$, and a pair of right and left eigenvectors $x_2=[0\ 0\ 1\ 0]^T$ and $y_2=[0\ 0\ 0\ 1]^T$ corresponding to $\lambda_2=-1$. Notice that $\langle x_1, y_1\rangle = \langle x_2, y_2\rangle = 0$, showing that conditions (ii) and (iii) of Theorem 1 are not satisfied and thus A does not have any reducing subspaces of dimension $k=1$; neither does any matrix similar to A. A similar argument applies to the case $k = 3$.

The special case of $\mathcal{M} = \mathcal{L}^{\perp}$ in condition (i) of Theorem 1 is treated in the following result.

Corollary 1. Let $A\in\mathbf{C}^{n\times n}$ and $1\le k < n$. The following are equivalent:

(i) there exists a subspace $A\in\mathbf{C}^n$ of dimension k such that $(\mathcal{L}, \mathcal{L}^{\perp})$ is a reducing pair for A;

(ii) there exists decomposable $x\in\mathbf{C}^{\binom{n}{k}}$ such that for all $s\in\mathbf{C}$, x is a common eigenvector of $(A + sI)^{(k)}$ and $(A^* + sI)^{(k)}$;

(iii) there exists decomposable $x\in\mathbf{C}^{\binom{n}{k}}$ and $\hat{s}\in\mathbf{C}$ such that $(A+\hat{s}I)$ is nonsingular and x is a common eigenvector of $(A + \hat{s}I)^{(k)}$ and $(A^* + \hat{s}I)^{(k)}$.

Moreover, when either of these conditions hold, x in (ii) and (iii) is a Grassmann representative of $\mathcal{L}$.

It should be noted that the above corollary also follows from [6, Theorem 2.2] by recalling that $(\mathcal{L}, \mathcal{L}^{\perp})$ is a reducing pair for A if and only if $\mathcal{L}$ is a common invariant subspace of A and A^*.

Theorem 1 provides a reducibility criterion for a single matrix in $\mathbf{C}^{n\times n}$. It can be extended to the case of simultaneous reduction of a pair of matrices as follows. Let $A,B\in\mathbf{C}^{n\times n}$ and $x, y\in\mathbf{C}^n$. We shall say that (x, y) is *a common pair of right and left eigenvectors of A and B* if x is a common right eigenvector of A and B and y is a common left eigenvector of A and B.

Theorem 2. Let $A, B\in\mathbf{C}^{n\times n}$ and $1\le k < n$. The following are equivalent:

(i) there exist subspaces $\mathcal{L}, \mathcal{M}\subset\mathbf{C}^n$ of dimensions k and $n-k$, respectively, such that $(\mathcal{L}, \mathcal{M})$ is a common reducing pair for A and B;

(ii) for all $s\in\mathbf{C}$, $(A + sI)^{(k)}$ and $(B + sI)^{(k)})$ have a common pair (x, y) of right and left eigenvectors $(x, y) \in \boldsymbol{C}^{\binom{n}{k}}$ that are decomposable and satisfy $\langle x, y\rangle \neq 0$;

(iii) there exists $\hat{s} \in \boldsymbol{C}$ such that $(A + \hat{s}I)$ and $(B + \hat{s}I)$ are nonsingular, and such that $(A + \hat{s}I)^{(k)}$, $(B + \hat{s}I)^{(k)})$ have a common pair (x, y) of right and left eigenvectors $x, y \in \boldsymbol{C}^{\binom{n}{k}}$ that are decomposable and satisfy $\langle x, y\rangle \neq 0$.

Moreover, when either of these conditions hold, x and y in (ii)} and (iii) are *Grassmann representatives of* $\mathcal{L}$ *and* $\mathcal{M}^{\perp}$, respectively.

The proof of Theorem 2 is similar to that of Theorem 1. It is also easily seen that the above result can be extended to the case of any number of matrices having a common pair of reducing subspaces. As in Corollary 1, we can now obtain a criterion for simultaneous reducibility of A and B by orthogonal complements.

Corollary 2. Let $A \in \boldsymbol{C}^{n\times n}$ and $1 \leq k < n$. The following are equivalent:

(i) there exists a subspace $\mathcal{L} \subset \boldsymbol{C}^n$ of dimension k such that $(\mathcal{L}, \mathcal{L}^{\perp})$ is a common reducing pair for A and B;

(ii) there exists decomposable $x \in \boldsymbol{C}^{\binom{n}{k}}$ such that for all $s \in \boldsymbol{C}$, x is a common eigenvector of $(A + sI)^{(k)}$, $(A^* + sI)^{(k)}$, $(B + sI)^{(k)}$, $(B^* + sI)^{(k)}$;

(iii) there exists decomposable $x \in \boldsymbol{C}^{\binom{n}{k}}$ and $\hat{s} \in \boldsymbol{C}$ such that $(A + \hat{s}I)$ and $(B + \hat{s}I)$ are nonsingular and x is a common eigenvector of $(A + \hat{s}I)^{(k)}$, $(A^* + \hat{s}I)^{(k)}$, $(B + \hat{s}I)^{(k)}$ and $(B^* + \hat{s}I)^{(k)}$.

Moreover, when either of these conditions hold, x in (ii) and (iii) is a *Grassmann representative of* $\mathcal{L}$.

4. Illustrative example

The next example illustrates the applicability of Theorem 2 for finding out a common reducing pair of subspaces. We will use the following criterion for the existence of a common eigenvector among two matrices.

Theorem 3 [5]. Let $X, Y \in \boldsymbol{C}^{p\times p}$ and

$$K(X, Y) = \sum_{m,l=1}^{p-1} [X^m, Y^l]^* [X^m, Y^l]$$

where $[X^m, Y^l]$ denotes the commutator $X^m Y^l - Y^l X^m$. Then X and Y have a common eigenvector if and only if K is not invertible.

Example 2. Let us consider whether

$$A = \begin{bmatrix} 0.5 & -6 & -6 & -1.5 \\ 0.5 & 3 & 1 & -0.5 \\ -0.5 & 0 & 2 & 0.5 \\ -1.5 & -6 & -6 & 0.5 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 2.5 & -3 & -4 & 0.5 \\ 0.5 & -2 & -4 & -0.5 \\ -0.5 & 0 & 2 & 0.5 \\ 0.5 & -3 & -2 & 2.5 \end{bmatrix}$$

have a common reducing pair of subspaces of dimension $k = 2$. For that purpose, recall Theorem 2 and in particular its third clause. The spectrum of A is $\{-1, 2, 3\}$ and the spectrum of B is $\{-2, 1, 3\}$. Thus A and B are nonsingular and we can take s = 0. Next compute the second compounds of A and B.

$$X = A^{(2)} = \begin{bmatrix} 4.5 & 3.5 & 0.5 & 12 & 7.5 & 4.5 \\ -3 & -2 & -0.5 & -12 & -3 & 0 \\ -12 & -12 & -2 & 0 & -12 & -12 \\ 1.5 & 1.5 & 0 & 6 & 1.5 & 1.5 \\ 1.5 & -1.5 & -0.5 & -12 & -1.5 & -2.5 \\ 3 & 6 & 0.5 & 12 & 3 & 4 \end{bmatrix},$$

$$Y = B^{(2)} = \begin{bmatrix} -3.5 & -8 & -1.5 & 4 & 2.5 & 4 \\ -1.5 & 3 & 1.5 & -6 & -1.5 & -3 \\ -6 & -3 & 6 & -6 & -6 & -9 \\ -1 & -1 & 0 & -4 & -1 & -1 \\ -0.5 & 1 & 1.5 & -8 & -6.5 & -11 \\ 1.5 & 6 & -1.5 & 6 & 1.5 & 6 \end{bmatrix}.$$

Referring to Theorem 3, the matrices $K = K(X, Y)$ and $K' = K(X^T, Y^T)$ are singular and so X, Y have common right and left eigenvectors. Note that if either K or K' were nonsingular, Theorem 2 would imply that A and B do not have a common reducing pair of subspaces.

Using Matlab's routine, we find that

$$\text{Nul}\,(X + 3I) = \text{span}\,\{x\}, \text{ where } x = [-1\ 0\ 0\ 0\ 1\ 0]^T;$$

notice that $Yx = -6x$ and thus x is a common right eigenvector of X, Y. Similarly, we see that

$$\text{Nul}\,(Y^T + 6I) = \text{span}\{y\}, \text{ where } y = [-1\ -1\ 0\ 0\ 1\ 1]^T;$$

notice that $X^T y = -3y$ and thus y is a common left eigenvector of X, Y.

Next we examine the decomposability of x, y. The quadratic Plucker's relations for decomposability can be used in this instance (see [4, Vol. II, 4.1, Definition 1.1]). For example, $[x_1, \ldots, x_6]^T \in \mathbf{C}^{\binom{4}{2}}$ is decomposable if and only if $x_1x_6 - x_2x_5 + x_3x_4 = 0$. It follows that x, y are decomposable. In fact, $x = \alpha_1 \wedge \alpha_2$ and $y = \beta_1 \wedge \beta_2$, where

$$\alpha_1 = [1\ 1\ 0\ 1]^T,\ \alpha_2 = [1\ -1\ 0\ 1]^T,\ \beta_1 = [0\ 1\ 1\ 0]^T,\ \beta_2 = [1\ 0\ 0\ 1]^T.$$

Notice that $\langle x, y\rangle \neq 0$. Hence, letting

$$\mathcal{L} = \text{span}\{\alpha_1, \alpha_2\} \text{ and } \mathcal{M} = (\text{span}\{\beta_1, \beta_2\})^{\perp},$$

by Theorem 2, we have that $(\mathcal{L}, \mathcal{M})$ is a common reducing pair for A and B. Indeed, if

$$L = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix},$$

where the first two columns of L have been computed to be a basis for $\mathcal{M}$ as defined above, we obtain the following simultaneous reductions of A and B:

$$L^{-1}AL = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -5 & 4 \end{bmatrix} \quad \text{and} \quad L^{-1}BL = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

In conclusion, the main results of the paper are necessary and sufficient conditions for the existence of non-trivial complementary subspaces which are reducing subspaces for a single matrix (Theorem 1) and for a pair of matrices (Theorem 2). Our study utilizes notions from multilinear algebra and is motivated by the numerous applications of the reducibility problem in different areas and especialy in the area of linear control systems theory. The above example is an illustration of the reducibilty criterion in finding out a pair of common reducing subspaces for two matrices and computing bases vectors of these subspaces. The results and their implementation also rise the questions of whether or not a subspace of k-th Grassmann space over $\boldsymbol{C}^n$ contains a nonzero decomposable vector and, on the other hand, how to find the factors of a decomposable vector. These issues are of particular interest from both theoretical and computational point of view and are discussed in more details in [6].

References

1. George, A, K. D. Ikramov. Common invariant subspaces of two matrices. – Linear Algebra and Its Applications. **287**, 1999, 171-179.
2. Gohberg, I., P. Lancaster, L. Rodman. Invariant Subspaces of Matrices with Applications. Wiley- Interscience, 1986.
3. Horn, R. A., C. R. Johnson. Matrix Analysis. Cambridge, Cambridge Univ. Press, 1985.
4. Marcus, M. Finite Dimensional Multilinear Algebra.Vol. 1 and 2. Marcel Dekker, 1973-1975.
5. Shemesh, D. Common eigenvectors of two matrices. – Linear Algebra and Its Applications. **62**, 1984, 11-18.
6. Tsatsomeros, M. Matrices with a common non-trivial invariant subspace. – Linear Algebra and Its Applications. **322**, 2001, 51-59.

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Владимир Монов[1], *Майкъл Цацомерос*[2]

[1] *Институт по информационни технологии, 1113 София*
E-mail: vmonov@iit.bas.bg
[2] *Mathematics Department, Washington State University, Pullman, WA 99164-3113, USA*
E-mail: tsat@math.wsu.edu

(Резюме)

Изследва се проблемът за съществуването и характеристиките на нетривиални редуциращи подпространства, като се използват някои основни средства на полилинейната алгебра. Получен е критерий за редуцируемостта на една матрица, който е разширен и за случая на едновременно редуциране на две или повече матрици.