BULGARIAN ACADEMY OF SCIENCES

CYBERNETICS AND INFORMATION TECHNOLOGIES • Volume **5**, No 2

Sofia • 2005

Hilbert Transform Relations

...Each continuous problem (differential equation) has many discrete approximations (difference equations).

Gilbert Strang (SIAM Review, Vol. 41, 1999, No 1, 135-147)

Bozhan Zhechev

Institute of Computer and Communication Systems, 1113 Sofia

Abstract : *In this paper the Hilbert transform different cases – for continuous, periodic and discrete signals are analyzed. The main attention is paid to the properties of the discrete cyclic transform. The eigenvectors and eigenvalues of this transform, projectors onto the region of the values, pseudoinverse endomorphism, and connections with another variants are found. The properties of the magnitude response of the different Hilbert's filters are demonstrated.*

Keywords: *digital signal processing, inverse filtering, bandpass signals, single-sideband modulation, image processing, discrete Fourier Transform (DFT), fast transforms, invariant spaces, pseudoinverse.*

I. Introduction

The Hilbert transform (or more correctly endomorphism) κ is applied in many areas: generating of single-sideband signals, inverse filtering, image processing, speech processing, radiolocation, compressing and etc. [1, 2, 3].

A purpose of this paper is to represent κ completely – when signals are defined on the set of the real numbers R, integer numbers Z, one-dimensional torus $\mathsf{R}|\mathsf{Z}$ and complete residue system modulo n, $\mathsf{Z}|n\mathsf{Z}$ [8]. This approach gives possibilities to obtain the basic properties that are difficult to be analyzed separately.

II. Continuous case (signals on R)

Let signals' domain be the real line R**.** The function $1/t$ is not summable in the vicinity of the point $t = 0$, but it's well known [4] that if φ has limited region of support and is at least one time differentiable at the beginning of the co-ordinates, this integral exists:

(1) $$(\text{v.p.}\frac{1}{t} \mid \varphi(t)) = \text{v.p.}\int_{-\infty}^{\infty} \frac{\varphi(t)}{t} dt \,.$$

Here v.p. symbols denote Cauchy principal value of the integral that follows. This integral defines linear continuous form of , therefore v.p. $1/t$ is a distribution (or generalized function) [4]. These distributions repeatedly are applied in quantum mechanics:

(2) $$\delta^{+} = \frac{\delta}{2} + \frac{1}{j2\pi}\text{v.p.}\frac{1}{\nu}, \quad \delta^{-} = \frac{\delta}{2} - \frac{1}{j2\pi}\text{v.p.}\frac{1}{\nu}.$$

They are the Fourier transforms of the unit step $Y(t)$ [4] and its mirror towards the ordinate axis $Y(-t) = \sigma Y(t)$ (the operator σ "reverses direction of the time", and δ is Dirac delta function). If F is the Fourier operator ($\omega = 2\pi\nu$, ν – frequency), it is well-known that

(3) $$F\left(\frac{1}{\pi t}\right) = -j\,\text{sgn}(\nu), \qquad \text{sgn}(\nu) = \begin{cases} 1, & \nu > 0 \\ 0, & \nu = 0 \\ -1, & \nu < 0 \end{cases},$$

$$Y(t) = \frac{1+\sigma}{2}Y(t) + \frac{1-\sigma}{2}Y(t) = \frac{1}{2}(1+\text{sgn}(t)).$$

A direct prove of the first equation could be done applying Lobachevski integral [5], or "periodizating" $1/t$:

(4) $$I(a) = \int_{-\infty}^{\infty} \frac{\sin(at)}{t} dt = \text{sgn}(a)\int_{-\infty}^{\infty} \frac{\sin(t)}{t} dt = \pi\,\text{sgn}(a),$$
$$F\left(\frac{1}{\pi t}\right) = \int_{-\infty}^{\infty} \frac{e^{-j2\pi\nu t}}{\pi t} dt = \frac{1}{j\pi}\int_{-\infty}^{\infty} \frac{\sin(2\pi\nu t)}{t} dt = \frac{1}{j\pi} I(2\pi\nu) = -j\,\text{sgn}(\nu).$$

Hilbert transform κ can be defined as a convolution of the signal $x(t)$ and v.p. $1/\pi t$.

(5) $$\kappa(x(t)) = \text{v.p.}\frac{1}{\pi t} x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau.$$

The Paley-Wiener condition [6] is necessary and sufficient for existing of Hilbert transform, and the reverse transform is given by $-\kappa$.

III. Continuous case (signals on T = R/Z)

Let $x(t)$ is such a function, that $x(t) = x(t+T)$, where T is a real number – the so-called period. The number 0 is always period; a number opposed to the period is period too and the sum of two periods is again a period. Thus the periods are some subgroup of the additive group of the real numbers R (see the definitions at the end of the paper). It is the so-called group of the periods. If $x(t)$ is continuous function its group of periods is closed subgroup of R**.** But there exist only three closed subgroups of R**:**

1. The subgroup, reduced to 0. A function without periods T different of 0 is aperiodic.

2. A whole group R**;** a function that has as a period every real number T is a constant.

3. The set of multiples kT_0, $T_0>0$, k – integer from Z.

If the group of periods belongs to one of the latest two cases, $x(t)$ is periodic; the number T_0 of the third one is so called main period of $x(t)$.

Let Γ is a circle with a center O and length T in the plane tOx. Every function $x(t)$ on Γ can be connected with a function $\tilde{x}$ on R**,** if $\tilde{x}(t) = x(M)$, where M is a point of Γ with a curvilinear abscissa $s = t$. The beginning of the reference is the point A of Γ on the axis Ot and direction is counter-clockwise.

The function $\tilde{x}$ is periodic with period T. And vice versa, if $\tilde{x}$ is periodic function on R with period T, it can be received with previous procedure from one and only function x. The mapping $x \to \tilde{x}$ is an isomorphism between the functions on Γ and R**.** One of the reasons for introducing of the periodic functions is that the functions on the trigonometric circle can be considered as functions of the angle θ with a period of 2π [4].

Let φ is a function on R**,** that could be made "periodical" in this way:

(6)
$$\tilde{\Phi}(t) = \sum_{l=-\infty}^{\infty} \varphi(t + lT).$$

If this functional series converges (for instance if j is with limited support), $\tilde{\Phi}$ will be periodic function with period T. When j $(t) = 1/t$ and this equilateral hyperbola in (6) is "coiled" on the unit circle, it will be uniformly convergent series:

(7)
$$\tilde{\Phi}(t) = \frac{1}{t} + \sum_{l<0} \frac{1}{t + 2\pi l} + \sum_{l>0} \frac{1}{t + 2\pi l} = \frac{1}{t} + \sum_{0<l} \frac{2t}{t^2 - 4\pi^2 l^2}.$$

The next identity can be proved by induction [7, p. 317]:

(8)
$$\operatorname{ctg}(t) = \frac{1}{2^n} \sum_{0 \le l < 2^n} \operatorname{ctg}\left(\frac{t + l\pi}{2^n}\right) =$$
$$= \frac{1}{2^n}\left(\left(\operatorname{ctg}\left(\frac{t}{2^n}\right) - \operatorname{tg}\left(\frac{t}{2^n}\right)\right) + \sum_{1 \le l < 2^{n-1}} \left(\operatorname{ctg}\left(\frac{t + l\pi}{2^n}\right) + \operatorname{ctg}\left(\frac{t - l\pi}{2^n}\right)\right)\right).$$

For $n \to \infty$ the first term of the second row of (8) converges to $1/t$, and the l-th term – to

$$\frac{2t}{t^2 - l^2\pi^2}.$$

Hence the function series in (7) is expansion of $(\operatorname{ctg}(t/2))/2$.

Another proof of this expansion (Euler's expansion) can be found in [5]. We have from here for the Hilbert transform of a periodical function $x(t)$ with period 2p:

$$\kappa(x(t)) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(t)}{t-\tau}dt = \sum_{k=-\infty}^{\infty}\int_{2\pi k}^{2\pi(k+1)}\frac{x(t)}{\pi(t-\tau)}dt =$$

(9)

$$= \sum_{k=-\infty}^{\infty}\int_{0}^{2\pi}\frac{x(\tau+2\pi k)}{\pi((t-\tau)-2\pi k)}dt = \frac{1}{2\pi}\int_{0}^{2\pi}x(\tau)\mathrm{ctg}\left(\frac{t-\tau}{2}\right)dt.$$

This result can be found in [1, p.781; 3, p. 618]. From (4) and (5) and the basic property of the Fourier operator – to transform convolution into algebraic multiplication [4], and from

$$F(\cos(t)) = \frac{1}{2}(\delta(\nu-1)+\delta(\nu+1)),$$

$$F(\sin(t)) = \frac{1}{2j}(\delta(\nu-1)-\delta(\nu+1))$$

(here $d(n)$ is a Dirac delta function [4]), follows:

(10a) $$k(\cos(t)) = \sin(t).$$

This result can be obtained from (6) too, because $\cos(t)$ is a periodic function with main period $2p$. It can be found too, that

(10b) $$k(\sin(t)) = -\cos(t).$$

These well-known and often-applied formulae – (10a) and (10b), are the most important relations of the Hilbert endomorphism (in this case k acts as an integral operator). They refer to every pair {cos($2pnt$), sin($2pnt$)}, n № 0. From them follow many interesting results. Hilbert transform connects real and imaginary part of the frequency response of a causal system, gain and phase of such a system, the envelope and phase of bandpass signals and etc. [1, 6].

IV. Discrete signals (signals on Z|nZ)

IV.1. General properties of the discrete (cyclic) endomorphism of Hilbert

Let's introduce these two operators (linear representations of the generators of a dihedral group $\mathbf{D}_n$) [8]:

$$\rho_n = [\delta_{k-1,l}],\ \sigma_n = [\delta_{k,n-l}],$$

$$k, l = 0, 1, ..., n-1 \pmod{n}$$

(11) $$\rho_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \sigma_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

These are respectively the right-shift operator ρ and sign operator σ for the signals on Z|nZ. The last set can be presented as vertexes of the inscribed – in circle Γ (with length $T = n$) regular n-polygon, received after "coiling" on it of R, and therefore of Z too. In that way one can construct the class of the discrete periodic functions. For the continuous case when $t \in \mathrm{R}$ (or in discrete case, when $t \in \mathrm{Z}$) these are the automorphisms:

$$\rho\colon x(t) \to x(t-1), \quad \sigma\colon x(t) \to x(-t).$$

In (11) $\delta_{l,k}$ is the Kronecker's symbol. Lets the dimension of the signals ("vectors") space n is an even number, and the discrete delta (vector) of Dirac has the form:

$$\vec{\delta} = [1, 0, 0, \ldots, 0]^{\mathrm{T}}.$$

In that case the sign vector – analog of sgn(v) from (4), has the form

$$\vec{s} = \frac{1}{\sqrt{n}}(1-\rho^{n/2}) \sum_{0<k<n/2} \rho^k \vec{\delta}. \tag{12}$$

For n = 8 this vector looks like this:

$$\vec{s} = \frac{1}{\sqrt{8}}[0, \quad 1, \quad 1, \quad 1, \quad 0, \quad -1, \quad -1, \quad -1]^{\mathrm{T}}.$$

If n is odd the "middle" zero will disappears. As in the continuous case, when $\sigma\, \mathrm{sgn}(v) = \mathrm{sgn}(-v) = -\mathrm{sgn}(v)$, this vector is odd, i.e. $\sigma\, \vec{s} = -\vec{s}$. The discrete Fourier operator has the form

$$F_n = \frac{1}{\sqrt{n}} \sum_{0 \le k,l < n} e^{-j\frac{2\pi k l}{n}} \rho^k \vec{\delta}\, \vec{\delta}^{\mathrm{T}} \rho^{-1}. \tag{13}$$

This operator is unitary, i.e. Hermitian-conjugated coincide with its inverse one [9]: $F_n F_n^*$=1. Therefore from (12) and of these dependences (they are demonstrated in [8]; modulation operator ? is defined after (15), and $\vec{f}_k$ is the k-th column of the discrete Fourier operator from (13)),

$$F\,\rho^k = ?^{-k}\, F; \quad F\vec{\delta} = \frac{\vec{1}}{\sqrt{n}}; \quad ?^{-k}\frac{\vec{1}}{\sqrt{n}} = \vec{f}_k,$$

one can obtain:

$$\vec{\kappa} = F_n^{\,*}(-j\,\vec{s}) = \frac{2}{n} \sum_{0 \le k < n/2} \mathrm{ctg}\,(\frac{\pi}{n}(2k-1))\, \rho^{2k-1}\, \vec{\delta},$$

(14)

$$\kappa(\rho) = \frac{2}{n} \sum_{0 \le k < n/2} ctg\,(\frac{\pi}{n}(2\,k-1))\, \rho^{2k-1}.$$

The first row of (14) is the impulse response, and the second one is the cyclic discrete endomorphism (system function) of Hilbert (an ideal cyclic Hilbert transformer or 90 degree phase shifter), that is antisymmetric and (anti-) commute with σ, i.e.

$$\kappa^{\mathrm{T}} = -\kappa = \sigma\,\kappa\,\sigma \Rightarrow \sigma\,\kappa = -\kappa\,\sigma \Rightarrow \sigma\,\kappa^2 = \kappa^2\,\sigma.$$

The magnitude response of this filter for n = 16 is given on Figs. 1 and 2. The vertical lines of the grid are drawn trough the points with abscises $\{\omega = 2\pi k/16\}_1^8$, for which the value is exactly 0 dB, and the magnitude response is pure imaginary. For the other frequencies deviations are big and a real component appears. The same behavior is, as it is shown in [13], of the analyzing filters of the Fast Fourier Transform (FFT). When designing of Hilbert transformers, the objective is an equiripple approximation of the sign-function [1, 3]. Applying of the (cyclic) FFT with such "bad" filters demonstrates, that this approach is not always obligatory.

It is of interest the endomorphism $\kappa^2(\rho)$, i.e. double applying of a Hilbert filter. Direct evaluating from (14) seems insuperable. The convolution of the impulse responses of two serial filters and the formula for the k-th co-ordinate of a convolution, derived in [8] gives us

$$\vec{\kappa}^2 = \vec{\kappa}.\vec{\kappa},$$

$$\kappa_k^2 = (\vec{\kappa}|\rho^k\ \sigma\ \vec{\kappa}) = (F\ \vec{\kappa}|F\ \rho^k\ \sigma\ \vec{\kappa})$$

(here $(\vec{a}|\vec{b})$ is an inner product of two vectors [9, 10]),

(15) $$\kappa^2(\rho) = -1 + \frac{2}{n}\sum_{0\le k<n/2}\rho^{2k} = -1 + \frac{1}{n}\vec{1}\,\vec{1}^{\mathrm{T}} + ?^{n/2}\frac{1}{n}\vec{1}\,\vec{1}^{\mathrm{T}}\,?^{-n/2}.$$

In previous equation $? = \mathrm{diag}(1,\ w,\ w^2,\dots,w^{n-1});\quad w = e^{j\frac{2\pi}{n}}$, is the mentioned before modulation operator [8] and $\vec{1}$ is the vector of all 1's. From it follows several important conclusions:

I) The operator $\kappa^2(\rho)$ – is orthogonal projector, as it is symmetric and

$$(-k^2(\rho))(-k^2(\rho)) = -k^2(\rho).$$

II) From (15) follows that $k^3(\rho) = -k(\rho)$; the two vectors $-\vec{1}$, $\mu^{n/2}\vec{1}$ are mapped from k into the zero vector (in frequency area it's obvious).

III) The generalize inverse endomorphism of k is $-k$; the pseudoinverse of k could be received if one can take into consideration I и II and that the pseudoinverse of an orthogonal projector P^+ co-inside with the same projector P [10]:

(16) $$\kappa^+ = (\kappa^{\mathrm{T}}\kappa)^+\kappa^{\mathrm{T}} = (-\kappa^2)^+(-\kappa) = (-\kappa^2)(-\kappa) = \kappa^3 = -\kappa.$$

IV) Let $\Re(\kappa) = \{\vec{z}{:}\vec{z} = \kappa(\vec{x})\}$ is the range of the endomorphism that is a linear sub-space [9, 10]. It's well known that $\kappa\kappa^+$ is an orthogonal projector into this sub-space, therefore this projector co-inside with $-k^2(\rho)$. Consequently we have for the dimensions of this subspace:

(17) $$\dim\Re(\kappa) = \dim\Re(-\kappa^2) = \mathrm{Tr}(-\kappa^2)\,.$$

From (15) follows, that the trace of the projector is

$$\mathrm{Tr}(-\kappa^2) = n - 2\,.$$

The kernel [9] $\mathrm{Ker}(\kappa)$ is not only the zero vector and this gives a reason k to be seen as an endomorphism but not as a transform, that will require it to be an automorphisms.

IV.2. Eigenvectors and eigenvalues of k

The Hilbert endomorphism is presented in the canonical bases $\{\rho^k\vec{\delta}\}_0^{n-1}$ from the circulant matrix (14), that's why its eigenvectors co-inside with the Fourier transforms columns [8]:

(18) $$\kappa\,F = F\,\Lambda;\quad F\,\kappa = -\Lambda\,F;\quad \kappa^{\mathrm{T}} = -\kappa;\quad F^{\mathrm{T}} = F.$$

Here Λ is a diagonal matrix of the eigenvalues. But the Fourier transform of the impulse response of κ (from (14)) has by definition the form:

(19) $$F\,\vec{\kappa} = F\,\kappa(\rho)\,\vec{\delta} = -\Lambda\,F\,\vec{\delta} = -\Lambda\frac{\vec{1}}{\sqrt{n}} = -j\,\vec{s}\,.$$

Therefore the diagonal matrix of the eigenvalues has the form

(20) $$\Lambda = \mathrm{diag}(\,j\,\vec{s}\,) = \mathrm{diag}(0,\ j,\ j,\ldots,j,\ 0,-j,-j,\ldots,-j)..$$

The endomorphism κ is an antisymmetric endomorphism, and hence it has pure imaginary eigenvalues and if l is an eigenvalue, than eigenvalue is $-l$.

When n is even, the zero and the $n/2$ rows of Λ have 0 and when n is odd the middle 0 drops out. Let $F = C - jS$, and these vectors are columns of $C, S(\sigma C = C;\ \sigma S = -S)$ respectively:

$$\{\vec{c}_k, \vec{s}_k;\ k = 0,1,\ldots,n-1\};\ \{\vec{c}_k = \vec{c}_{n-k};\quad \vec{s}_k = -\vec{s}_{n-k}, k = 1,\ldots,n/2-1\}.$$

Then from (18) and (20) it follows:

(21) $$\kappa\,\vec{c}_k = \vec{s}_k,\quad 0 \le k \le \frac{n}{2},$$
$$\kappa\,\vec{s}_k = -\vec{c}_k,\quad 0 < k < \frac{n}{2}.$$

It can be seen here, that κ rotates on 90° the pairs of orthogonal bases vectors $\{\vec{c}_k, \vec{s}_k\}$ of the two-dimensional subspaces of dihedral group D_n [8] (90-degree phase shifter) except the zero and (n/2)-th one-dimensional subspaces, that have basis respectively $\{\vec{c}_0, \vec{c}_{n/2}\}$ and are mapped into zero. The endomorphism κ rotates on 90° every vector with real co-ordinates $\vec{x}$:

$$(\vec{x}|\kappa\,\vec{x}) = (\kappa^T\,\vec{x}|\vec{x}) = (-\kappa\,\vec{x}|\vec{x}) = -(\vec{x}|\kappa\,\vec{x}) = 0.$$

From here $\vec{x} \perp \kappa\,\vec{x}$ and if $\vec{x}$ don’t have components from the kernel Ker(κ), this will be pure rotation. Otherwise besides rotation of 90°, the vector will be “shorted” because of its kernel components. If it is from the kernel, it will be mapped into the zero.

Equations (21) specify how does this rotation and “shortening” become.

It’s received from (19) and (20) the norm [9, p.330] of κ as the maximal (real) eigenvalue of the symmetric matrix $\kappa^T\kappa$:

$$\kappa^{\mathrm{T}}\kappa\,F = (-\kappa^2)F = F\,(-\Lambda^2),$$
$$\|\,\kappa\,\|^2 = \lambda_{\max}(\kappa^{\mathrm{T}}\kappa) = 1,$$

$$\|\,\kappa\,\vec{x}\,\| \le \|\,\kappa\,\|\,\|\,\vec{x}\,\|.$$

Therefore κ “shorten” the signals, and in the best case it saves their energy (norm).

The results of this part have many interesting applications. The complex filter $(1+j\kappa)$ forms the so-called analytic signal [1, 2, 3, 6]. It has, as follows from (19) and (20), only one-sided Fourier transform:

$$F(1+j\,\kappa)\,\vec{x} = (1-j\,\Lambda)\,F\,\vec{x} = 2\,\vec{Y}\circ\vec{X};\quad F\,\vec{x} = \vec{X}\,;$$
$$\vec{Y} = [1/2, 1, 1, \ldots, 1, 1/2, 0, 0, \ldots, 0].$$

Here $\vec{Y}$ is the discrete unit step; its zero and $n/2$ components are ? (the symbol "$\circ$" denotes componentwise multiplication of two vectors or Schur's multiplication). The filter 1 + jk extracts, in continuous case, the upper sideband, but in the discrete one this is "not entirely the same", mainly because of the form of $\vec{Y}$. It can be designed a this one filter:

$$\beta = \rho\,(1 \;+\; \kappa^2) + j\,\kappa; \quad \beta^2 = 1; \quad \beta^* = \beta.$$

The automorphisms β is an involution (its square is identity) and it is a Hermitian morphism i.e. coincide with its Hermitian-conjugated. In that case for the two orthogonal projectors $(1+\beta)/2$ and $(1–\beta)/2$ it will be true that:

$$F\left(\frac{1+\beta}{2}\right) = \text{diag}\;(\vec{1},\vec{0})\;F,$$
$$F\left(\frac{1-\beta}{2}\right) = \text{diag}\;(\vec{0},\vec{1})\;F.$$

The first projector "cuts off" the upper $n/2$ co-ordinates of the spectrum of a signal, and the second one – the lower $n/2$ co-ordinates. It can be shown that these two filters participate in constructing of the full recursive form of FFT [8, 13].

The real filter 1 + k, in contrast to k, is an automorphisms, i.e. there exists inverse one, which permits reconstruction of the input signal. The inverse filter is

$$(1+\kappa)^{-1} = \frac{1}{2}\,(\kappa^2 - \kappa + 2).$$

Every one of the orthogonal signals $\vec{x}$ and $k\,\vec{x}$ can be extracted with this filter from the mixture of them.

The most important property of the Fourier transform is, as it is well known, that it transforms convolution into multiplication and vice-versa – multiplication into convolution [4]. For the signals on $Z|nZ$ this property looks like:

$$F\,(\vec{x}.\vec{y}) = \sqrt{n}\;\; F\,\vec{x} \circ F\,\vec{y},$$
$$F\,(\vec{x} \circ \vec{y}) = \frac{1}{\sqrt{n}}\,F\,\vec{x}.F\,\vec{y}.$$

Let $\vec{c}_k,\vec{s}_k$ are the κ-th columns of C and S from (21). It's easy to be shown that

$$F\,\vec{c}_k = \frac{1}{2}\,(\rho^k + \rho^{-k})\,\vec{\delta}; \quad F\,\vec{s}_k = \frac{1}{2j}(\rho^k - \rho^{-k})\,\vec{\delta}; \quad (\rho^k\;\vec{\delta}).\vec{x} = \rho^k\;\vec{x}.$$

Applying of these relations and forming of the real signal

$$\vec{z} = \vec{c}_k \circ \vec{x} + \kappa\,(\vec{c}_k) \circ \kappa\,(\vec{x}) = \vec{c}_k \circ \vec{x} \;+\; \vec{s}_k \circ \kappa\,(\vec{x})\,,$$

gives ($\vec{Y}$ is the discrete unit step):

$$F\,\vec{z} = \frac{1}{\sqrt{n}}\;(\rho^k\,((\sigma\vec{Y}) \circ \vec{X}) + \rho^{-k}\,(\vec{Y} \circ \vec{X}))\,.$$

These formulae represent discrete variant of the well-known scheme of Hartley for modulation with a single sideband.

V. Discrete signals (signals on Z)

The Hilbert endomorphism could be obtained for signals on Z from the former case. The last row of (14) can be represent in the form (we assume that *n* is divisible of 4; this doesn't decrease generality):

$$\kappa(\rho) = \frac{2}{n} \sum_{0<k\le n/4} \operatorname{ctg}\left(\frac{\pi}{n}(2k-1)\right) \rho^{2k-1} + \frac{2}{n} \sum_{n/4<k\le n/2} \operatorname{ctg}\left(\frac{\pi}{n}(2k-1)\right) \rho^{2k-1}.$$

If we change in the second sum the variable $k \leftarrow (\frac{n}{2}+1-k)$, it can be obtained the form (one can see the noncausality and anti-symmetry of the Hilbert's filter):

$$\kappa(\rho) = \frac{2}{n} \sum_{0<k\le n/4} \operatorname{ctg}\left(\frac{\pi}{n}(2k-1)\right)\left(\rho^{2k-1} - \rho^{-(2k-1)}\right). \tag{22}$$

When *n* goes to infinity

$$\lim_{n\to\infty}\left(\frac{2}{n}\operatorname{ctg}\left(\frac{\pi}{n}(2k-1)\right)\right) = \frac{2}{(2k-1)\pi}. \tag{23}$$

We obtain the endomorphism of Hilbert for signals on **Z**:

$$\kappa(\rho) = \sum_{0<k<\infty} \frac{2}{(2k-1)\pi}\left(\rho^{2k-1} - \rho^{-(2k-1)}\right). \tag{24}$$

Here ρ gets the meaning of a right shift operator (delaying) defined after (15), where *t* is a number from Z.

This result could be obtained directly if we consider the expansion of the impulse response of the Hilbert filter (defined as a function on the group of the integer numbers Z) by the group of characters of Z, isomorphic of the group of the one-dimensional torus – a corollary of the so-called Pontryagin duality) [11, 12]. If the frequency response has by definition on the interval [–? , ?] (including fully the unit circle T, *w* = 2*pn*) the form

$$-j\ \operatorname{sgn}(v) = -j \begin{cases} 1, & 0 < v < 1/2, \\ 0, & v = 0, \\ -1, & -1/2 \le v < 0, \end{cases} \tag{25}$$

then for the *k*-th coefficient of the impulse response will be obtained

$$\kappa_k = \int_{-1/2}^{0} j\, e^{j2\pi v k} dv - \int_{0}^{1/2} j\, e^{j2\pi v k} dv = \begin{cases} \dfrac{2\sin^2(k\pi\pi 2)}{k\pi}, & k \ne 0, \\ 0, & k = 0. \end{cases} \tag{26}$$

This is the *k*-th coefficient of the expansion in (24). This very result is given in [1, p.792]. The problems begin from here for designing of appropriate filter with finite impulse response. On the Fig. 3 is given magnitude response of the filter from (24) with the first four terms of the series. The convergence of this series to the sgn-function is not uniform but mean square. Comparing of Figs. 1 and 3 shows that by equal number of terms the cyclic filter has smaller ripples. More over, it goes very accurately trough the unit (or 0 dB) into the points of sampling $\{\omega = 2\pi k / n\}_1^{n/2}$.

VI. Conclusion

This paper deals with a new approach to the different relations of the Hilbert transform. Trough introducing of the different regions of definition were revealed some sides of the relation "continuous-discrete" signals. Discrete cyclic Hilbert transform was analyzed. The eigenvectors and eigenvalues of this transform, pseudoinverse transform, projectors into the region of the values, was found.

It's shown that k rotates on 90° the invariant spaces of the dihedral group. The magnitude response of the cyclic Hilbert transformer, shown on the Fig.1 and Fig. 2 possess interesting properties. It is with big ripples but in the sampling points it's very accurate. The obtained properties of this transform permits to be analyzed connections of the (cyclic!) Fast Fourier Transform with the Hilbert transform, that will be an object of another work.

Fig.1. Magnitude response; Eq. (22) has 4 terms (n = 16)

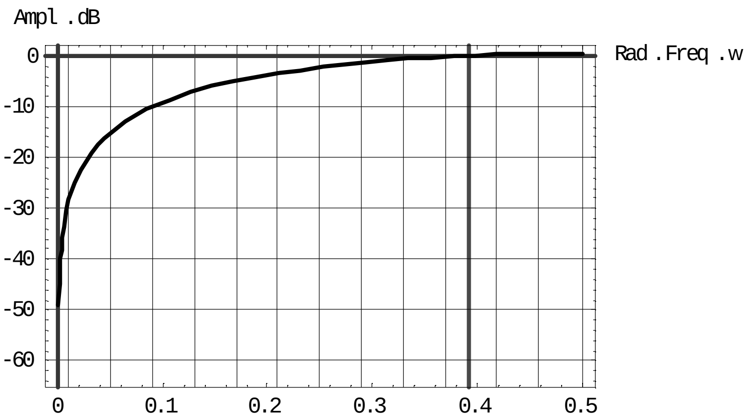

Fig.2. Magnitude response near 0; Eq. (22) (n = 16)

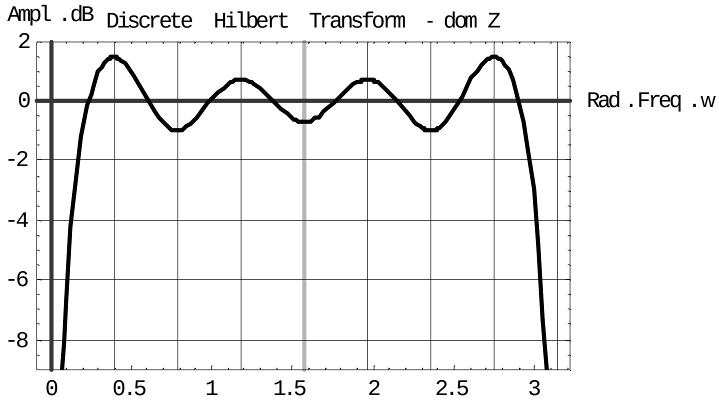

Fig. 3. Magnitude response; first 4 terms of Eq. (24)

Some definitions

The group G is a set G with binary operation $G \times G \to G$, *noted as* $(a,b) \to ab$ *and such, that:* ***1.*** *It is associative.* ***2.*** *Identity element u∈G exists, i.e. ua = a = au for every a ∈ G.* ***3.*** *For every element a ∈ G an inverse element a' ∈ G exists, and aa' = u = a'a.*

If G and H are groups, the morphism φ:G → H of these groups is a function from G to H, which is morphism of their binary operations, i.e. φ(ab)=φ(a)φ(b) for all a,b∈G.

R e f e r e n c e s

1. O p p e n h e i m, A., R. S c h a f e r, J. B u c k. Discrete-Time Signal Processing. Prentice Hall, 2-nd ed., 1999.
2. P a p o u l i s, A. Systems and Transforms with Applications in Optic. McGraw-Hill, 1986.
3. P r o a k i s, J., D. M a n o l a k i s. Digital Signal Processing. Prentice-Hall, Int., 1996.
4. S c h w a r t z, L. Metodes Mathematiques Pour les Sciences Physiques. Paris, Herman, 1961.
5. Ф и х т е н г о л ь ц, Г. М. Курс диффереционального и интегрального вычисления. Москва, Наука,1968.
6. M a s o n, S. J., H. J. Z i m m e r m a n. Electronic Circuits, Signals and Systems. John Wiley & Sons Inc., 1960.
7. G r a h a m, R., D. K n u t h, O. P a t a s h n i k. Concrete Mathematics a Foundation for Computer Science. Addison-Westley, 1994.
8. Z h e c h e v, B. Invariant spaces and fast transforms. – In: IEEE Trans. on Circuits and Syst. II: Analog and Digital Signal Proc., February 1999, p. 216.
9. S t r a n g, G. Linear Algebra and its Applications. Academic Press, 1976.
10. A l b e r t, A. Regression and the Moor-Penrose Pseudoinverse. Academic Press, 1972.
11. S e r r e, J.-P. Representations Lineaires des Groupes Finis. Hermann, Paris, 1967.
12. M o r r i s, S.A. Pontryagin Duality and the Structure of Locally Compact Abelian Groups. London, New York, Cambridge University Press, 1977.
13. Z h e c h e v, B. Fast transforms – analysis. – In: Int. Conference Automatics and Informatics, Sofia, Bulgaria, 31 May 2001, I-69, I-72.