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Models for Weighted Aggregation of Fuzzy Relations to Multicriteria Decision Making Problems

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Abstract. *Aggregation of fuzzy relations on alternatives with the help of aggregation operators is considered in two cases:*

– *the weighted coefficients of the criteria are not present in the mathematical formula of the aggregation operators;*

– *a fuzzy preference relation between the criteria importance is given.*

The main result consists in proving the properties of the aggregated relation in dependence with the individual relations' properties. These properties give a possibility to decide the ranking, choice or clustering problems by fuzzy multicriteria decision making.

Keywords: *fuzzy preference relations, properties of fuzzy relations, t-norms, composition of fuzzy relation, aggregation operators, decision making.*

1. Introduction

Most multicriteria decision making models have been developed using mainly fuzzy relations corresponding to the several criteria. One of the problems in this case is connected with the aggregation of these relations in such a way, that the union relation should be the fuzzy one, providing a possibility to decide the choice, the ranking or the clustering problems. A purposeful approach for uniting individual fuzzy relations is to use the aggregation procedures that realize the idea of compensation and compromise between conflicting criteria, when compensation is allowed. The aggregation operators may perform these procedures, e.g. Weighted Mean [2, 35, 36], Weighted Geometric [3], Weighted MaxMin and Weighted MinMax [11, 26] operators. Several operators with parameters are introduced, e.g. MaxMin, MinAvg, Gamma operators [38], Generalized Mean operator [7, 12, 27].

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A very good overview of the aggregation operators, by presenting the characteristics, the advantages and disadvantages of each operator and the relationships between them, is available in [7, 12].

The fuzzy relations are the basic concept in the following multicriteria decision making problem in this investigation. Let $A = \{a_1,...,a_n\}$ be the finite set of alternatives evaluated by several fuzzy criteria $K = \{k_1,...,k_m\}$, i.e. these criteria give fuzzy relations $\{R_1, R_2,..., R_m\}$ between the alternatives. When the cardinality n of A is small, the relations may be represented by the $n \times n$ matrices $R_k = \|r_{ij}^k\|$, where $r_{ij}^k = \mu_k(a_i, a_j), i, j = 1,...,n, k = 1,...,m$, $\mu_k: A\times A \to [0, 1]$ is the membership function of the relation R_k and r_{ij}^k is the membership degree to R_k by comparison of the couple of the alternatives a_i and a_j by the criterion k_k.

The setting problem is to obtain the aggregated relation between the alternatives uniting the fuzzy relations by the individual criteria taking into account the importance of the criteria. Aggregation of fuzzy relations on the alternatives with the help of aggregation operators is considered in two cases here:

– the weighted coefficients of the criteria are not present in the mathematical formula of the aggregation operators;

– these coefficients are given as fuzzy preference relation between the criteria importance.

2. Weighted transformations in aggregation operators

Let the set of weighted coefficients (weights) of the criteria $K = \{k_1, ..., k_m\}$ be $W = \{w_1, ..., w_m\}$. The weights are present in the mathematical models of some aggregation operators. How one can use these weights in cases, when they are not presented in the aggregation operator's formula? The general procedure to include these weights in the aggregation process uses some transformation of the values $r_{ij}^k = \mu_k(a_i, a_j)$, $i, j = 1, ..., n$, $k = 1, ..., m$, under the importance degree w_k to generate a new value. This transformation can be made with the help of a function with required properties. Examples of the transformation function include: the minimum operator [35], an exponential function [34], any t-norm operator [20, 38], a linguistic quantifier [4].

The idea of Yager considered in [35] is developed here. Let "Agg" denotes an aggregation operator. Each of the membership degrees may be transformed using weights as follows

$$g(w_i, \mu_i(a,b)) = \tilde{\mu}_i(a,b),\ a, b \in A,\ i = 1,...,m,$$

and then the weighted aggregation is obtained as:

$$\mathrm{Agg}(\tilde{\mu}_1(a,b),...,\tilde{\mu}_m(a,b)) = \mu^w(a,b).$$

The function g satisfies the following properties [35]:

- $x > y \to g(w, x) \geq g(w, y)$;
- $g(w, x)$ is monotone in w;
- $g(0, x) = \mathrm{id}$, $g(1, x) = x$,

where the identity element, "id", is such that it doesn't change the aggregated value if we add it to our aggregation. The form of g depends on the type of aggregation being performed. In performing the Min aggregation there are elements with small values that play the most significant role in this type of aggregation. One way to reduce the effect of elements with low importance is to transform them into values closer to one. That's way a class of functions that can be used for the inclusion of weights in the Min aggregation is:

$$g(w_i, \mu_i(a,b)) = S(1-w_i, \mu_i(a,b)),$$

where S is a t-conorm, and then

$$\min\{\tilde{\mu}_1(a,b),..., \tilde{\mu}_m(a,b)\} = \min\{S(1-w_1, \mu_1(a,b)),..., S(1-w_m, \mu_m(a,b))\}.$$

In this case if $w_i = 0$, then $S(1-w_i, \mu_i(a,b)) = S(1, \mu_i(a,b)) = 1$ and this element plays no pole in the Min aggregation.

In performing the Max aggregation the transformation

$$g(w_i, \mu_i(a,b)) = T(w_i, \mu_i(a,b))$$

may be used, where T is a t-norm. If $w_i = 0$, then $T(w_i, \mu_i(a,b)) = 0$ and the element plays no role in the aggregation.

Relations that are of interest in this research are fuzzy ones pertaining to the similarity or likeness of the alternatives and these ordering alternatives. The aggregation of these fuzzy relations is made in order to get the union relation as a fuzzy one solving the ranking, choice or clustering of the set of alternatives. In [19] the relationship between the properties of the individual relations and the ones of the aggregated relation are investigated to decide the above problems with the help of chosen aggregation operator.

The purpose of the following investigation is to prove that the above weighted transformations preserve or do not preserve the properties of the fuzzy relations and how to use these transformations in the mathematical models of the aggregation operators.

2.1. Properties of the weighted transformations connected with fuzzy relations

The properties of the following weighted transformations will be proved:

(1) $$\tilde{\mu}(a,b) = g(w, \mu(a,b)) = T(w, \mu(a,b)),$$

(2) $$\tilde{\tilde{\mu}}(a,b) = g(w, \mu(a,b)) = S(1-w, \mu(a,b)),$$

where T and S are t-norm and t-conorm, respectively. The following more common t-norms and their dual t-conorms will be considered here:

Table 1

	t-norm	t-conorm
Drastic	$T_0(x,y) = \begin{cases} x & \text{if } y=1 \\ y & \text{if } x=1 \\ 0 & \text{anywhere else} \end{cases}$	$S_0(x,y) = \begin{cases} x & \text{if } y=0 \\ y & \text{if } x=0 \\ 1 & \text{anywhere else} \end{cases}$
Lukasiewicz	$T_1(x,y) = \max(0, x+y-1)$	$S_1(x,y) = \min(1, x+y)$
	$T_2(x,y) = \dfrac{xy}{2-(x+y-xy)}$	$S_2(x,y) = \dfrac{x+y}{1+xy}$
Probabilistic	$T_3(x,y) = xy$	$S_3(x,y) = x+y-xy$
	$T_4(x,y) = \dfrac{xy}{x+y-xy}$	$S_4(x,y) = \dfrac{x+y-2xy}{1-xy}$
Min -Max	$T_5(x,y) = \min(x,y)$	$S_5(x,y) = \max(x,y)$

As it is well known the linear orderings of the above *t*-norms and *t*-conorms are:

$$T_0 \le T_1 \le T_2 \le T_3 \le T_4 \le T_5;\ S_5 \le S_4 \le S_3 \le S_2 \le S_1 \le S_0.$$

The investigations whether the weighted transformations (1), (2), i.e. the *t*-norms and *t*-conorms from Table 1 preserve or do not preserve the properties of the fuzzy relations are represented in Table 2 and proved in [20, 22]. The fuzzy relations' properties defined in [17] and their abbreviations are:

r (reflexivity) $\mu(a, a) = 1, \forall a \in A,$

s (symmetry) $\mu(a, b) = \mu(b, a), \forall a, b \in A,$

perf antis (perfect antisymmetry) if $\mu(a, b) > 0$ then $\mu(b, a) = 0, \forall a, b \in A,$

rec (reciprocity) $\mu(a,b) + \mu(b,a) = 1, \forall a,b \in A,$

m asy (moderate asymmetry) $\mu(a,b) + \mu(b,a) \le 1, \forall a,b \in A,$

w asy (weak asymmetry) $\min(\mu(a,b), \mu(b,a)) \le 0.5, \forall a,b \in A,$

m cmp (moderate comparability) $\mu(a,b) + \mu(b,a) \ge 1, \forall a, b \in A,$

w cmp (weak comparability) $\max(\mu(a,b), \mu(b,a)) \ge 0.5, \forall a, b \in A,$

max-min tr (max-min transitivity) $\mu(a, c) \ge \min(\mu(a, b), \mu(b, c)), \forall a, b\ c \in A,$

m tr (moderate transitivity) if $\mu(a, b) \ge 0.5$, $\mu(b, c) \ge 0.5$, then $\mu(a, c) \ge \min(\mu(a, b), \mu(b, c)), \forall a, b, c \in A,$

w tr (weak transitivity) if $\mu(a, b) \ge 0.5$, $\mu(b, c) \ge 0.5$ then $\mu(a,c) \ge 0.5, \forall a, b, c \in A,$

max-Δ tr (max-Δ transitivity) $\mu(a, c) \ge \max(0, \mu(a, b) + \mu(b, c) - 1), \forall a, b, c \in A,$

The definitions of some special fuzzy relation are [9, 17, 30]:

D1 (similarity relation) $\Leftrightarrow \mathbf{r} \wedge \mathbf{s} \wedge \mathbf{max-min}$ tr;

D2 (likeness relation) $\Leftrightarrow \mathbf{r} \wedge \mathbf{s} \wedge \mathbf{max} - \Delta$ tr;

D3 (weak F-weak order) $\Leftrightarrow$ **w comp** $\wedge$ **w** tr;

D4 (moderate F-weak order) $\Leftrightarrow$ **m comp** $\wedge$ **m** tr;

D5 (fuzzy total ordering) $\Leftrightarrow$ **rec** $\wedge$ **w** tr;

D6 (fuzzy partial order relation) $\Leftrightarrow \mathbf{r} \wedge$ **perf antis** $\wedge\ \mathbf{max} - \Delta$ tr;

D7 (fuzzy preorder) $\Leftrightarrow \mathbf{r} \wedge \mathbf{max} - \Delta$ tr;.

D8 (fuzzy linear ordering)a fuzzy partial ordering such that for $\forall a,b \in A$ if $a \ne b$, either $\mu(a,b) > 0$ or $\mu(b,a) > 0$;

D9 (nonfuzzy linear ordering) $\Leftrightarrow$ any α-cut of a fuzzy linear ordering.

In the first column of Table 2 are presented the *t*-norms and *t*-conorms used in (1) and (2) and in the first row are given the proved properties in [20, 22].

Table 2. Relationship between t-norms and t-conorms and fuzzy relation' properties Y – preserve, N – not preserve

Norm	**r**	**s**	**perf antis**	**rec**	**m asy**	**w asy**	**m cmp**	**w cmp**	**max-min tr**	**m tr**	**w tr**	**max-Δ tr**
T	N	Y	Y	**m asy**	Y	Y	N	N	Y	Y	N	Y
S	Y	Y	N	**m cmp**	N	N	Y	Y	Y	Y	Y	N*

* Exception: *t*-conorms S_3, S_4, S_5, preserve the max-Δ transitivity.

A generalization of t-norms and t-conorms from Table 1 is the parameterized t-norms and t-conorms. Some of the above t-norms and t-conorms can be obtained by varying the parameters in these parameterized norms. Three families of parameterized t-norms and t-conorms: the Hamacher, the Yager and the Weber-Sugeno ones are the aim of this investigation. The relationships between these norms and the ones from Table 1 and their properties are investigated in [21].

Hamacher *t*-norm. The Hamacher t-norms are defined for $\gamma \geq 0$ by:

$$T_{\mathrm{H}}(x,y) = \frac{xy}{\gamma + (1-\gamma)(x+y-xy)} = \frac{xy}{\gamma + (1-\gamma)S_3(x,y)}.$$

This function is decreasing, i.e. $T_{\mathrm{H}} \to 0$ when $\gamma \to \infty$. The following particular cases can be noted:

– $\gamma = 0 \Rightarrow T_{\mathrm{H}} = T_4$; $\gamma = 1 \Rightarrow T_{\mathrm{H}} = T_3$; $\gamma = 2 \Rightarrow T_{\mathrm{H}} = T_2$;

– $T_{\mathrm{H}} \geq T_1 \geq T_0$,

– $\min(x,y) > T_{\mathrm{H}} \geq 0$.

The Hamacher t-conorms are defined for $\gamma \geq 0$ by:

$$S_{\mathrm{H}}(x,y) = \frac{x+y-(2-\gamma)xy}{\gamma + (1-\gamma)(1-xy)}.$$

This function is increasing, i.e. $S_{\mathrm{H}} \to 1$ when $\gamma \to \infty$. The following particular cases can be noted:

– $\gamma = 0 \Rightarrow S_{\mathrm{H}} = S_4$; $\gamma = 1 \Rightarrow S_{\mathrm{H}} = S_3$; $\gamma = 2 \Rightarrow S_{\mathrm{H}} = S_2$;

– $S_{\mathrm{H}} \leq S_1 \leq S_0$,

– $\max(x,y) < S_{\mathrm{H}} \leq 1$.

Yager *t*-norm. The Yager t-norms are defined for p>0 by:

$$T_{\mathrm{Y}}(x,y) = \max\left(1 - \left[(1-x)^p + (1-y)^p\right]^{1/p}, 0\right).$$

This function is increasing and the following particular cases are interesting [7]:

– if $p \to \infty, T_{\mathrm{Y}} \to \min(x,y)$;

– if $p = 1.709$, then a t-norm is not the probabilistic t-norm, but it is the closest one in the Yager family in sense that it equals the product on the boundary $T(x,1) = x, T(x,0) = 0$) and in (0.5, 0.5);

– if p=1 then $T_{\mathrm{Y}} = T_1$;

– if $p \to 0$ then the drastic t-norm cannot be obtain;

– for $0 < p \leq 1$ the Yager t-norms do not have a bijection and for p>1 have a bijection.

The Yager t-conorms are defined for p>0 by:

$$S_{\mathrm{Y}}(x,y) = \min\left((x^p + y^p)^{1/p}, 1\right).$$

This function is decreasing and the following particular cases will be noted:

– if $p \to \infty, S_{\mathrm{Y}} \to \max(x, y)$;

– if p = 1.709, then a *t*-conorm is not the probabilistic *t*-norm, but it is the closest one in the Yager family in sense that it equals the product on the boundary ($S(x,0) = x, S(x,1) = 1$) and in (0.5, 0.5);

– if p = 1, then the Lukasiewicz *t*-conorm is obtained, i.e. $S_{\mathrm{Y}} = S_1$;

– if $p \to 0$, then the drastic *t*-conorm cannot be obtain;

– for $0 < p \le 1$ the Yager *t*-norms do not have a bijection and for p>1 have a bijection.

Weber–Sugeno *t*-norm. The Weber-Sugeno *t*-norms are defined for $\lambda_t > -1$ by:

$$T_{\mathrm{W}}(x, y) = \max\left(\frac{x + y - 1 + \lambda_t xy}{1 + \lambda_t}, 0\right).$$

This function is decreasing and the following particular cases are interesting [7]:

– this *t*-norm cannot generalize (even approach) $T_5(x, y) = \min(x, y)$;

– if $\lambda_t \to -1$ then it cannot obtain the drastic *t*-norm;

– if $-1 < \lambda_t \le 0$ then T_{W} do not have a bijection;

– if λ_t=0 then T_{W}= T_1; if $\lambda_t \to \infty$ then $T_{\mathrm{W}} \to T_3$.

The Weber–Sugeno *t*-conorms are defined for $\lambda_s \ge -1$ by

$$S_{\mathrm{W}}(x, y) = \min(x + y + \lambda_s xy, 1).$$

It can be noted that the duality between T_{W} and S_{W} is satisfied if $\lambda_s = \dfrac{\lambda_t}{1 + \lambda_t}$.

This function is increasing and the following particular cases are interesting:

– this t-conorm cannot generalize (even approach) $S_5(x, y) = \max(x, y)$;

– if λ_t=–1 then ; if λ_t=0 then S_{W}= S_1; if $\lambda_t \to \infty$ then $S_{\mathrm{W}} \to S_0$.

The relationship between *t*-norms and *t*-conorms and fuzzy relation' properties are common for all *t*-norms and *t*-conorms. It is proved in [21], that the preservation of the max-Δ transitivity for each parameterized *t*-norm and *t*-conorm is performed by different conditions. This is represented in Table 3.

Table 3. Preservation of max-Δ transitivity by parametrized *t*-norms and *t*-conorms (Y – preserve, N– not preserve)

Norm	*t*-norms	*t*-conorms
Hamacher	Y	Y only for $\gamma \le 1$
Yager	Y	Y only for $p \to \infty$
Weber–Sugeno	Y	Y only for $-1 \le \lambda < 0$

2.2. Properties of the weighted transformed aggregation operators

As it was already said, the aggregation operators in the mathematical models of which the weights do not present will be considered here. These operators have the following form:

$\mathrm{Agg}(\mu_1(a,b),...,\mu_m(a,b))$, where $\mu_i(a,b), i=1,...,m,$ is the membership degree to the fuzzy relation $R_i, i=1,...,m$, by comparison of the couple of the alternatives a and b by the criterion k_i. Then, the weighted transformation of given aggregation operator will be: $\mu^w(a,b)=\mathrm{Agg}(\mu_i^w(a,b),...,\mu_m^w(a,b))$, where

(3) $$\mu_i^w(a,b)=g(h(w_i),\mu_i(a,b)), i=1,...,m,$$

with weight coefficient $w_i, i=1,...,m$, the transformation function g, which may be either t-norm and then (3) is (1) or t-conorm and then (3) is (2). The dependence between the properties of $\mu^w(a,b), \forall a,b\in A$ and $\mu_i^w(a,b), i=1,...,m$, is the purpose of the research. Tables 2 and 3 are used to show when the t-norms and t-conorms preserve the fuzzy relations' properties. The weighted transformations of the following well-known aggregation operators, according to the reasons made above, can be: Convex linear compensatory operators, Exponential compensatory operators, Generalized mean operator.

Convex linear compensatory operators

The mathematical model of the weighted transformation of these operators is:

$$\mu^w(a,b)=(1-\gamma)T(\tilde{\tilde{\mu}}_1(a,b),...,\tilde{\tilde{\mu}}_m(a,b))+\gamma\, S(\tilde{\mu}_1(a,b),...,\tilde{\mu}_m(a,b)), \gamma\in[0,1],$$

where $\tilde{\mu}_i(a,b)=T(w_i,\mu_i(a,b))$ and $\tilde{\tilde{\mu}}_i(a,b)=S(1-w_i,\mu_i(a,b))$, according to (1) and (2).

The representatives of this group of operators are:

a) Max-Min aggregation operator with the membership function

$$\mu^w(a,b)=\gamma\max_i\{\tilde{\mu}_i(a,b)\}+(1-\gamma)\min_i\{\tilde{\tilde{\mu}}_i(a,b)\}.$$

b) Min-Avg aggregation operator with the membership function

$$\mu^w(a,b)=\gamma\sum_{i=1}^m \tilde{\mu}_i(a,b)+(1-\gamma)\min_i\{\tilde{\tilde{\mu}}_i(a,b)\}.$$

Exponential compensatory operators

The mathematical model of the weighted transformation of these operators is:

$$\mu^w(a,b)=[T(\tilde{\tilde{\mu}}_1(a,b),...,\tilde{\tilde{\mu}}_m(a,b))]^{1-\gamma}\bullet[S(\tilde{\mu}_1(a,b),...,\tilde{\mu}_m(a,b))]^{\gamma}, \gamma\in[0,1];$$

a) The well-known **Gamma operator** belongs to this group:

$$\mu^w(a,b)=\left[\prod_{i=1}^m \tilde{\tilde{\mu}}_i(a,b)\right]^{1-\gamma}\left[1-\prod_{i=1}^m(1-\tilde{\mu}_i(a,b))\right]^{\gamma}=A(a,b)^{1-\gamma}(1-B(a,b))^{\gamma}.$$

Generalized mean operator

The mathematical model of this operator is:

$$\mu(a,b) = \left[\frac{1}{m}\sum_{i=1}^{m}\mu_i(a,b)^{\sigma}\right]^{\frac{1}{\sigma}}, \text{ where } \sigma \neq 0 \text{ is a real number.}$$

It reduces to the harmonic, geometric, arithmetic and quadratic mean operators, when $\sigma = -1, \sigma \to 0, \sigma = 1, \sigma = 2,$ respectively. When $\sigma \to -\infty$, the Generalized mean operator approaches the Min operator, and when $\sigma \to \infty$, it approaches the Max operator.

a) The weighted transformation of the **Min operator** is considered in [35] and it is

$$\mu^w(a,b) = \min_i \{S(1 - w_i, \mu_i(a,b)\}.$$

b) The following weighted transformation will be considered for the **Harmonic operator**

$$\mu(a,b) = \begin{cases} \dfrac{m}{\sum\limits_{i=1}^{m}\dfrac{1}{S(1-w_i,\mu_i(a,b))}} = \dfrac{m}{B(a,b)}, \text{ if } \mu_i(a,b) \neq 0, \forall a,b \in A, \\ 0, \text{ otherwise,} \end{cases}.$$

because, if w_i=0 then $S(1 - w_i, \mu_i(a,b)) = 1 \neq 0$ and this element decreases the value of $\mu^w(a,b)$, while if w_i=0 and e.g., $\tilde{\mu}_i(a,b) = T(w_i, \mu_i(a,b)), i = 1,..., m,$ then $T(w_i, \mu_i(a,b)) = 0$ and therefore $\mu^w(a,b) = 0$.

c) The **Geometric mean operator** may be transformed using weights as follows

$$\mu^w(a,b) = \sqrt{\prod_{i=1}^{m} S(w_i, \mu_i(a,b))},$$

because, if w_i=0, then $S(w_i, \mu_i(a,b)) \geq S(0, \mu_i(a,b)) = \mu_i(a,b)$, i.e. this is the smallest value of $S(w_i, \mu_i(a,b))$. If the other possibilities for $\tilde{\mu}_i(a,b)$ are considered, one can see that:

– if $\tilde{\mu}_i(a,b) = S(1 - w_i, \mu_i(a,b))$ and w_i=0, then $S(1 - w_i, \mu_i(a,b)) = 1$, therefore the value of $\tilde{\mu}_i(a,b)$ is the greatest;

– if $\tilde{\mu}_i(a,b) = T(w_i, \mu_i(a,b))$ and w_i=0, then $\tilde{\mu}_i(a,b) = 0$ and hence $\mu^w(a,b) = 0$;

– if $\tilde{\mu}_i(a,b) = T(1 - w_i, \mu_i(a,b))$ and w_i=0, then $\tilde{\mu}_i(a,b) = \mu_i(a,b)$, but $T(1 - w_i, \mu_i(a,b)) \leq T(1, \mu_i(a,b)) = \mu_i(a,b)$.

d) The **Arithmetic mean operator** may be transformed as follows:

$$\mu^w(a,b) = \sum_{i=1}^{m} T(w_i, \mu_i(a,b)),$$

because, if w_i=0, then $T(w_i, \mu_i(a,b)) = 0$ and this element plays no role in the sum.

e) The **Quadratic mean operator** may be transformed as follows:

$$\mu^w(a,b) = \sqrt{\frac{1}{m}\sum_{i=1}^{m} T(w_i, \mu_i(a,b))^2},$$

because, if w_i=0, then $T(w_i, \mu_i(a,b)) = 0$ and this element plays no role in the sum.

f) The weighted transformation of the **Max operator** is considered in [35] and it is:

$$\mu^w(a,b) = \max_i\{T(w_i, \mu_i(a,b)\}.$$

The dependencies between the properties of the aggregated relation R and the properties of the individual relations R_i for each of the above operators are presented in Table 4. The proofs of these relationships are given in [20, 22]. The Table 4 contains the implications: if R_i, i=1, ..., m, belong to **Pi** or **Di** then R belongs to **Pj** or **Dk**, where **Pi** denotes a class of fuzzy relations which posses the property **Pi** (r – reflexivity, s – symmetry, t – max-min transitivity) and **Di** denotes a class of fuzzy relations (**D1**– similary relation, **D2** – likeness relation, **D3** – fuzzy preorder). The notation $r \wedge t \wedge (4) \rightarrow D3$, e.g. denotes – if the individual relations R_i, $i = 1, ..., m$, are reflexive, max-min transitive and the condition (4) is hold then the aggregated relation by given operator is weak F-weak order, where

(4) $$0 \le w_i \le 1 \text{ and } \sum_{i=1}^{m} w_i = 1,$$

(5) $$0 \le w_i \le 1 \text{ and } \max_i w_i = 1$$

and one of the conditions (6) must be hold for the Gamma operator $\gamma = 0$;

(6) $$\gamma \in (0,1] \quad \text{and } 0 < \mu(a,b) \le 0.5, \forall a,b \in A;$$

$$\gamma \in (0,1], \min\{(1 - B(a,b))\} \ge [\max\{(1 - B(a,b))\}]^2, \forall a,b \in A.$$

Table 4. Connection between the properties of and R

Norm	$R_i \rightarrow R$	$R_i \rightarrow R$
Max-Min	**D1** $\wedge (5) \rightarrow$ **D2**	**r** $\wedge$ **t** $\wedge (5) \rightarrow$ **D3**
Min-Avg	**D1** $\wedge (4) \rightarrow$ **D2**	**r** $\wedge$ **t** $\wedge (4) \rightarrow$ **D3**
Gamma	**D1** $\wedge (6) \wedge (5) \rightarrow$ **D2**	**r** $\wedge$ **t** $\wedge (6) \wedge (5) \rightarrow$ **D3**
Min	**D1** $\rightarrow$ **D2**	**r** $\wedge$ **t** $\rightarrow$ **D3**
Harm	**D1** $\rightarrow$ **D2**	**r** $\wedge$ **t** $\rightarrow$ **D3**
Geom	**D1** $\rightarrow$ **D2**	**r** $\wedge$ **t** $\rightarrow$ **D3**
Arith	**D1** $\wedge (4) \rightarrow$ **D2**	**r** $\wedge$ **t** $\wedge (4) \rightarrow$ **D3**
Quadr		**s** $\rightarrow$ **s**
Max	**r** $\wedge (7) \rightarrow$ **r**	**s** $\rightarrow$ **s**

3. Aggregation by fuzzy preference relation between the criteria importance

The fuzzy preference relations are the basic concept in the following multicriteria decision making problem here. Let $A = \{a_1, ..., a_n\}$ be the finite set of alternatives evaluated by several fuzzy criteria $K = \{k_1, ..., k_m\}$, i.e. these criteria give fuzzy preference relations R_1, R_2, ..., R_m between the alternatives. When the cardinality n of A is small, the preference relations may be represented by the $n \times n$ matrices $R_k = \left\| r_{ij}^k \right\|$, where $r_{ij}^k = \mu_k(a_i, a_j), i, j = 1, ..., n, \ k = 1, ..., m,$ is the membership function of the relation R_k and r_{ij}^k is the preference degree of the alternative a_i over a_j by the criterion k_k; $r_{ij}^k = 0.5$ indicates indifference between a_i and a_j, $r_{ij}^k = 1$ indicates that a_i is absolutely preferred to a_j, and $r_{ij}^k > 0.5$ indicates that a_i is preferred to a_j by the k-th criterion. In this case, the preference matrices R_k, k=1, ..., m, are usually assumed to be additive reciprocal, i.e.

$$r_{ij}^k + r_{ji}^k = 1, \ i, j, = 1, ..., n.$$

A fuzzy preference relation W between the criteria is given as well, i.e. the couples of criteria are compared according to their importance. Let $W = \left\| w(k_i, k_j) \right\| = \left\| w_{ij} \right\|, i, j = 1, ..., m,$ where $w(k_i, k_j)$ be the preference degree of the criterion k_i over k_j. The setting problem is to obtain the preference relation between the alternatives uniting the fuzzy relations by the individual criteria taking into account the relation between the importance of the criteria. The aim is to use the whole information given above to the final stage of the problem solution, without transforming the relation W into some weighted coefficients.

To make a consistent choice or to rank the alternatives from the “best” to the “worst” one, when assuming fuzzy preference relations, a set of properties to be satisfied has been suggested. The consistency in this case has a direct effect on the ranking results of the final decision. The investigations on the consistency of fuzzy preference relations are made in [13, 15, 31, 33]. The study of consistency is associated with the concept of transitivity [13]. Let $\mu : A \times A \rightarrow [0,1]$ is a membership function of a fuzzy relation and $a, b, c \in A$. Some of transitivity properties are:

- Max-min transitivity [9, 38]: $\mu(a, c) \geq \min(\mu(a, b), \mu(b, c))$;
- Max-max transitivity [29]: $\mu(a, c) \geq \max(\mu(a, b), \mu(b, c))$;
- Restricted max-min transitivity or moderate transitivity [29]:
  $\mu(a, b) \geq 0.5, \ \mu(b, c) \geq 0.5 \Rightarrow \mu(a, c) \geq \min(\mu(a, b), \mu(b, c))$;
- Restricted max-max transitivity [29]:
  $\mu(a, b) \geq 0.5, \ \mu(b, c) \geq 0.5 \Rightarrow \mu(a, c) \geq \min(\mu(a, b), \mu(b, c))$;
- Additive transitivity [29]: $\mu(a, c) = \mu(a, b) + \mu(b, c) - 0.5$.

Characterizations and comparisons between these transitivity properties are suggested in [13]. The additive transitivity is a stronger property than restricted max-max one, which is a stronger concept than the restricted max-min transitivity, but a weaker property than max-max transitivity. The latter property is a stronger one than max-min transitivity, which is a stronger property than restricted max-min transitivity. Methods for constructing fuzzy preference relations from preference data are described in [1, 13, 10, 32]. Applying these methods it is possible to get consistency of the fuzzy preference relations and in such a way, to keep away from inconsistent solutions in the decision making processes.

The dependences between the properties of the aggregated fuzzy preference relation and the ones of the individual relations for this case are investigated in [5, 6, 8, 14, 16, 19, 22, 25].

3.1. Method with new fuzzy preference relation

Let $\mu_i(a, b)$ be the membership degree of the comparison of the alternatives $a,b \in A$ to the fuzzy preference relation R_i. Taking into account the relation W, a new fuzzy relation R_{ij}, $i, j = 1, \ldots, m$, between R_i and R_j, $R_i \neq R_j$, with the following membership degrees is suggested:

$$r_{ij}(a,b) = \begin{cases} 0.5 & \text{if } a = b, \\ S(T(w_{ij}, \mu_i(a,b)), T(w_{ij}, \mu_j(a,b))) & \text{if } a \neq b, \end{cases} \tag{7}$$

where T is a t-norm and S is a corresponding t-conorm.

The core idea used in (7) is that the comparison operator “pessimistically” combines the two relations to obtain measures of match which can be after that “optimistically” united in an overall result. Thus, if a t-norm provides the “pessimistic” combination, a t-conorm is a suitable generalization of the concept of “optimistic” union. As $R_{ij} = R_{ji}$, the number k of the new relations will be equal to the combinations of two elements over m, i.e. $k = \dfrac{m(m-1)}{1.2}$. Aggregation operators uniting these k relations can be used after that to obtain the aggregation fuzzy relation giving a possibility to decide the choice or ranking problems. The transitivity is one of the most important properties concerning preferences. The purpose of the following investigation is to prove that the fuzzy relation with membership degrees (7) preserve, or do not preserve the transitivity property of the individual fuzzy relations. The examples show that the relation (7) does not preserve the max-min transitivity. But this kind of transitivity is a very strong property imposed on a fuzzy relation according to [37]. Z a d e h [37] suggested several useful definitions of transitivity, which are compared in [30]. The weakest of all these definitions is the max-Δ transitivity. It is shown that this is the most suitable notion of transitivity for fuzzy ordering. That’s why the following proposition proves the conditions under which the relation (7) is a max-Δ transitive one. This proposition is proved for the different t-norms and the corresponding t-conorms from Table 1.

Proposition 3.1 [23]**.** The relation (7) is max-Δ transitive if the relations R_i, $i = 1, \ldots, m$, are max-min transitive and the relation W is additive reciprocal.

3.2. Aggregation of fuzzy preference relations by composition

One attempt to use the composition of two relations in an aggregation procedure is investigated here. If the composition possesses some properties required for solving the problems of ranking or choice, then it may be used in such procedures. This will give one practical application of the composition.

Definition [18]**.** Let X and Y be fuzzy relations in $A\times A$ and let T be a t-norm. The composition $X\circ Y$ of these relations with respect to T is the fuzzy relation on $A\times A$ with membership function

(8) $$\mu(a_i,a_j)=\mu_{X\circ Y}(a_i,a_j)=\max_k\{T(\mu_X(a_i,a_k),\mu_Y(a_k,a_j))\},\ i,j,k=1,...,n.$$

When T = min then the composition is a max-min one. When $T = xy$, then it is a max-product composition. $X\circ Y$ can be obtained as the matrix product of X and Y. It has to be taken into account that $X\circ Y\neq Y\circ X$.

Let $X=\|x_{ij}\|$ and $\mathrm{Y}=\|y_{ij}\|$, $i,j=1,...,n$, be fuzzy relations in $A\times A$, where x_{ij},y_{ij} be the membership degrees of the comparison of the alternatives $a_i,a_j\in A$ to the fuzzy preference relations X and Y, respectively.

It has to be investigated what kind of transitivity must possess both relations to have their composition some transitivity properties. The examples show that the composition does not preserve the transitivity property, but it transforms the additive and max-max transitivity into the max-Δ one (see propositions 3.2.1, 3.2.2). Besides the composition of two restricted max-min or max-max transitivity relations is not always a max-Δ transitive relation.

Proposition 3.2.1 [24]**.** If two fuzzy preference relations are additively transitive, then the composition of these relations is max-Δ transitive.

The additive transitivity does not imply the max-max one, for that reason the following proposition is suggested.

Proposition 3.2.2. If two fuzzy preference relations are max-max transitive, then the composition of these relations is max-Δ transitive.

P r o o f: Let $X=\|x_{ij}\|$ and $\mathrm{Y}=\|y_{ij}\|$, $i,j=1,...,n$ be max-max transitive relations, i.e. $x_{ij}\geq\max(x_{ik},x_{kj})$ and $y_{ij}\geq\max(y_{ik},y_{kj})$, $\forall i,j,k$.

If $Z=X\circ Y$, $Z=\|z_{ij}\|$, it has to be proved that

$$z_{ij}\geq\max(0,z_{ik}+z_{kj}-1),\ k=1,...,n.$$

A) *Max-min composition*

Let

$$z_{ik}=\max_s\{\min(x_{is},y_{sk})\}=\min(x_{ik_1},y_{k_1k}),\ s=1,...,k_1,...,n,$$
$$z_{kj}=\max_s\{\min(x_{ks},y_{sj})\}=\min(x_{kk_2},y_{k_2j}),\ s=1,...,k_2,...,n,$$
$$z_{ij}=\max_s\{\min(x_{is},y_{sj})\}=\min(x_{ik_3},y_{k_3j}),\ s=1,...,k_3,...,n,$$

and let, $z_{ik}=x_{ik_1}$, $z_{kj}=x_{kk_2}$, $z_{ij}=x_{ik_3}$ i.e. it has to be proved that

$$x_{ik_3}\geq\max(0,x_{ik_1}+x_{kk_2}-1).$$

But $x_{ik_3} \geq \max(x_{ik_1}, x_{k_1k_3}) \geq x_{ik_1} \geq x_{ik_1} x_{kk_2} \geq \max(0, x_{ik_1} + x_{kk_2} - 1)$. The proofs of the other variants of the minimum values of z_{ik}, z_{kj}, z_{ij} are reduced to this case.

B) *Max-product composition*

Let

$$z_{ik} = \max_s \{x_{is} . y_{sk}\} = x_{ik_1} . y_{k_1k},\ s = 1, \ldots, k_1, \ldots, n,$$
$$z_{kj} = \max_s \{x_{ks} . y_{sj}\} = x_{kk_2} . y_{k_2j},\ s = 1, \ldots, k_2, \ldots, n,$$
$$z_{ij} = \max_s \{x_{is} . y_{sj}\} = x_{ik_3} . y_{k_3j},\ s = 1, \ldots, k_3, \ldots, n,$$

and let $z_{ik} = x_{ik_1} . y_{k_1k}$, $z_{kj} = x_{kk_2} . y_{k_2j}$, $z_{ij} = x_{ik_3} . y_{k_3j}$, i.e. it has to be proved that

$$x_{ik_3} . y_{k_3j} \geq \max(0, x_{ik_1} . y_{k_1k} + x_{kk_2} . y_{k_2j} - 1) .$$

But for $\forall s$, $x_{ik_3} \geq \max(x_{is}, x_{sk_3})$, i.e. $x_{ik_3} \geq x_{ik_1}$, analogically $y_{k_3j} \geq y_{k_2j}$. Therefore

$$x_{ik_3} . y_{k_3j} \geq x_{ik_1} . y_{k_2j} \geq \max(0, x_{ik_1} + y_{k_2j} - 1) \geq \max(0, x_{ik_1} . y_{k_1k} + x_{kk_2} . y_{k_2j} - 1) .$$

The numerical examples show that the composition of two fuzzy relations does not preserve the other properties of the fuzzy relations.

Taking into account the relation W, the fuzzy relation R from (7) between X and Y, $X \neq Y$, with the following membership degrees, is suggested:

$$r_{ij} = \begin{cases} 0.5 & \text{if } a_i = a_j, \\ S(T(w^1, z_{ij}^1), T(w^2, z_{ij}^2)) & \text{if } a_i \neq a_j, \end{cases} \tag{9}$$

where $Z^1 = X \circ Y = \left\| z_{ij}^1 \right\|$, $z_{ij}^1 = \max_k \{T(x_{ik}, y_{kj})\}$,

$$Z^2 = Y \circ X = \left\| z_{ij}^2 \right\|,\ z_{ij}^2 = \max_k \{T(y_{ik}, x_{kj})\}, \quad k = 1, 2, \ldots, n,$$

$w^1 = w(k_X, k_Y)$, $w^2 = w(k_Y, k_X)$ are the preference degrees of the criterion with a relation X over Y and Y over X, respectively, T is a t-norm and S is a corresponding t-conorm.

The main idea used in (9) is that the composition of two relations compares the preference degrees of the i-th alternative to all alternatives from the first relation with the preference degrees of the all alternatives to the j-th alternative from the second relation and vise versa, because the operation composition is not commutative. Then taking into account the relation W, i.e. that w^1 and w^2 are the preference degrees of the relation X over Y and Y over X, respectively, a comparison operator is used that "pessimistically" combines the relations W and Z^1, W and Z^2 to obtain measures of match which can be after that "optimistically" united in an overall result. Aggregation operators [19] uniting these k relations can be used after that to obtain the aggregated fuzzy relation giving a possibility to decide the choice or ranking problems. The following proposition is essential for this purpose.

Proposition 3.2.3 [24]**.** If the relations Z^1, Z^2 are max-Δ transitive ones and the relation W is additive reciprocal, then the relation (9) is max-Δ transitive for the couple of t-norms (T = min, S = max**)** and (T = xy, S = x + y – xy**).**

4. Concluding remarks

Weighted aggregations are important in decision making problems where one has multiple criteria to consider and where the outcome is to be judged in terms of criteria which are not equally important for the decision maker. It is more realistic to use fuzzy relations because they appear as a more convenient and adequate form for representing the relationship between alternatives then crisp relations. The fuzzy relations may model situations, whenever interactions between the alternatives are not exactly determined. Besides that, they reflect the interests of the experts or the decision-maker.

Aggregation of fuzzy relations on the alternatives with the help of aggregation operators is the basic concept in the two models considered above.

The first model uses weighted transformations of aggregation operators for uniting of fuzzy relations. The proved connections between the properties of the individual fuzzy relations and the ones of the aggregated relations given in the table forms assist for solving the decision making problems.

In the second model a combination of t-norm and t-conorm is studied for obtaining a new fuzzy preference relation. This relation connects the individual fuzzy preference relations with the relation between the fuzzy criteria importance which evaluate the set of alternatives. It is proved, that this relation preserves under the defined conditions the property of transitivity.

Composition of two fuzzy preference relations usage in an aggregation procedure is investigated as well. It is proved, that the composition is max-Δ transitive, if both relations are additively or max-max transitive. It points out that the idea to use the composition to aggregate relations has a practical application. The new fuzzy preference relation is used after that. It is proved that the aggregated fuzzy preference relation preserves the max-Δ transitivity of the composition under the defined conditions.

The suggested models for aggregation use the whole information to the final step of problem solving and the proved properties give a possibility to decide the fuzzy multicriteria decision making problems.

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