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On Almost Complete Caps in PG(N, q)

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Abstract**:** *We propose the concepts of almost complete subset of an elliptic quadric in the projective space PG(3, q) and of almost complete cap in the space PG(N, q), $N \geq 3$, as generalizations of the concepts of almost complete subset of a conic and of almost complete arc in PG(2, q). Upper bounds of the smallest size of the introduced geometrical objects are obtained by probabilistic and algorithmic methods.*

Keywords: *Projective space, complete cap, complete arc, almost complete cap, almost complete arc.*

1. Introduction

Let PG(N, q) be the N-dimensional projective space over the Galois field F_q of order q. A cap in PG(N, q) is a set of points no three of which are collinear. An n-cap of PG(N, q) is complete if it is not contained in an (n + 1)-cap of PG(N, q). Caps in PG(2, q) are called also arcs. A point P of PG(N, q) is *covered* by a cap $\mathcal{K} \subset$ PG(N, q) if P lies on a bisecant of $\mathcal{K}$. The space PG(N, q) contains $\theta_{N,q} = \dfrac{q^{N+1}-1}{q-1}$ points.

An n-arc in PG(N, q) with $n > N+1$ is a set of n points such that no $N+1$ points belong to the same hyperplane of PG(N, q). An n-arc of PG(N, q) is complete if it is not contained in an (n + 1)-arc of PG(N, q). In PG(N, q) with $2 \leq N \leq q-2$, a normal rational curve is any (q + 1)-arc projectively equivalent to the arc $\{(1, t, t^2, \ldots, t^N): t \in F_q\} \cup \{(0, \ldots, 0, 1)\}$.

For an introduction to the spaces on finite fields, see [8-11] and the references therein.

The concept of *almost complete subset of a fixed irreducible conic in the plane* PG(2, q) is considered in [12], see also [3, 14] and the references therein.

Definition 1. In PG(2, q), an almost complete subset of a fixed irreducible conic is a proper subset of the conic covering all the points of PG(2, q) except for the remaining points of the conic and its nucleus if q is even.

Almost complete subsets of conics are useful for the classical problems of completeness of normal rational curves and extendability of generalized doubly-extended Reed-Solomon codes.

Let $t(q)$ be the smallest size of an almost complete subset of a conic in PG(2, q). Let an $[n, k, d]_q$ code be a q-ary linear code of length n, dimension k, and minimum distance d.

In [14] it is proved that under the condition

$$3 \le N \le q + 2 - t(q),$$

every normal rational curve in PG(N, q) is a complete (q + 1)-arc. (This assertion is equivalent to the following one: no $[q + 1, N + 1, q - N + 1]_q$ generalized doubly-extended Reed-Solomon code can be extended to a $[q + 2, N + 1, q - N + 2]_q$ Maximal Distance Separable (MDS) code [3].) In [3], the following upper bound is obtained:

$$t(q) < \sqrt{q(3\ln q + \ln\ln q + \ln 3)} + \sqrt{\frac{q}{3\ln q}} + 4 \sim \sqrt{3q\ln q}.$$

The concept of *almost complete arc in* PG(2, q) is considered in [15] where arcs of an infinite family $\mathcal{K}(q)$ are called almost complete if

$$\lim_{q\to\infty} \frac{\#\text{points not covered by } \mathcal{K}(q)}{\#\text{points of the plane PG}(2,q)} = 0. \qquad (1)$$

An almost complete subset of a conic is an almost complete arc as the number of points not covered by it is smaller than q.

Almost complete arcs are useful for investigations of upper bounds on the smallest size of saturating sets and complete arcs in PG(2, q).

In this work, we generalize both the aforementioned concepts.

Definition 2. (i) In PG(3, q), an Almost Complete subset of the elliptic Quadric $\mathcal{Q}$ (ACQ-subset, for short) is a proper subset of $\mathcal{Q}$ covering all the points of PG(3, q) except for the remaining points of $\mathcal{Q}$.

(ii) In PG(N, q), $N \ge 2$, a cap $\mathcal{K}$ is almost complete if the number of points not covered by $\mathcal{K}$ is not greater than $\theta_{N-1,q}$.

Note that if caps of Definition 2(ii) form an infinite family of caps $\mathcal{K}(q)$ in the spaces PG(N, q) with growing q then it holds that (cf. (1))

$$\lim_{q\to\infty} \frac{\#\text{points not covered by } \mathcal{K}(q)}{\#\text{points of the space PG}(2,q)} \le \frac{\theta_{N-1,q}}{\theta_{N,q}} = 0.$$

An ACQ-subset is an almost complete cap as the number of points not covered by it is smaller than $q^2 + 1$.

Let $d(q)$ be *the smallest size of an* **ACQ-subset** in PG(3, q).

Let $v(N, q)$ be *the smallest size of an almost complete cap* in PG(N, q).

This work is devoted to *upper bounds* on $d(q)$ and $v(N, q)$.

The main results of this work are presented in

Theorem 1. (i) In PG(3, q), for the smallest size of an ACQ-subset, we have

(2) $$d(q) \le (q+1)\sqrt{6\ln(q+1)} + 2q + 2 \sim q\sqrt{6\ln q}.$$

(ii) In PG(N, q), for the smallest size of an almost complete cap, it holds that

(3) $$v(N,q) \le \sqrt{2N\theta_{N-1,q}\ln q} + 1 \sim q^{\frac{N-1}{2}}\sqrt{2N\ln q}, \quad N \ge 2.$$

Moreover, an almost complete cap of size at most $\sqrt{2N\theta_{N-1,q}\ln q}+1$ can be constructed by a step-by-step greedy algorithm that in every step adds to the running cap a point providing the maximal possible (for the given step) number of new covered points.

One see that the bounds (2) and (3) asymptotically coincide with each other.

As far as it is known to the authors, ACQ-subsets and almost complete caps in PG(N, q), $N \ge 3$, are not considered in the literature. Therefore, it remains for us only to compare the bounds (2) and (3) with the known bounds on the smallest size $t_2(N, q)$ of a complete cap in PG(N, q). Of course, one should remember that these bounds are obtained for objects which are similar to the almost complete caps but not the same.

In [4], it is proved that

$$t_2(N, q) < cq^{\frac{N-1}{2}}\log^{300} q, \text{ with a constant } c \text{ independent of } q.$$

In [2], see also [6], under some probabilistic conjecture, it is shown that

(4) $$t_2(N,q) < \frac{1}{q-1}\sqrt{q^{N+1}(N+1)\ln q} + \frac{\sqrt{q^{N+1}}}{q-3} \sim q^{\frac{N-1}{2}}\sqrt{(N+1)\ln q}.$$

We see that $q^{\frac{N-1}{2}}\sqrt{2N\ln q}$ is essentially smaller than $cq^{\frac{N-1}{2}}\log^{300} q$. On the other side, the bound $q^{\frac{N-1}{2}}\sqrt{2N\ln q}$ (that is *proved rigorously*) is greater than the conjectural bound (4). So, the bounds of Theorem 1, obtained in this work, seem to be reasonable.

These new concepts and the methods of their investigation can be useful for bounds and constructions of small saturating sets and small complete caps, including a rigorous proof of the conjectural bound (4).

This paper is organized as follows. In Section 2, the bound (2) is proved by probabilistic methods. In Section 3, the bound (3) is obtained by an algorithmic approach.

Some results of this work were briefly presented in [7].

2. An upper bound on the smallest size of an almost complete subset of an elliptic quadric in PG(3, q)

Let $w > 0$ be a fixed integer. Let $\mathcal{Q}$ be an elliptic quadric in PG(3, q). Consider a random $(w+1)$-point subset $\mathcal{K}_{w+1} \subset \mathcal{Q}$. The total number of such subsets is $\binom{q^2+1}{w+1}$.

A fixed point A of PG(3, q) \ $\mathcal{Q}$ is covered by $\mathcal{K}_{w+1}$ if it belongs to a bisecant of $\mathcal{K}_{w+1}$. We denote by Prob($\diamond$) the probability of some event $\diamond$.

We estimate

$$\pi := \text{Prob}(A \text{ not covered by } \mathcal{K}_{w+1}),$$

as the ratio of the number of (w + 1)-point subsets of $\mathcal{Q}$ not covering A over the total number $\binom{q^2+1}{w+1}$ of subsets of $\mathcal{Q}$ with size (w + 1). A set $\mathcal{K}_{w+1}$ does not cover A if and only if every line through A contains at most one point of $\mathcal{K}_{w+1}$.

Through any point $A \in$ PG(3, q) \ $\mathcal{Q}$, there are $\frac{q(q-1)}{2}$ bisecants and q + 1 tangents of $\mathcal{Q}$ [9]. Every bisecant has two places to put a point of $\mathcal{K}_{w+1}$ while a tangent has the only one. For simplicity of presentation, we assume that a tangent also has two places to put a point of $\mathcal{K}_{w+1}$. (This will slightly worsen our estimates.) Therefore,

$$\pi < \frac{2^{w+1}\binom{q(q-1)/2+q+1}{w+1}}{\binom{q^2+1}{w+1}} = \frac{2^{w+1}\binom{(q^2+q+1)/2}{w+1}}{\binom{q^2+1}{w+1}},$$

where the numerator estimates from above the number of (w+1)-point subsets of $\mathcal{Q}$ not covering A. By straightforward calculations,

$$(5) \quad \pi < \frac{\left(q^2+q+2\right)\left(q^2+q\right)\left(q^2+q-2\right)...\left(q^2+q+2-2i\right)...\left(q^2+q+2-2w\right)}{\left(q^2+1\right)\left(q^2\right)\left(q^2-1\right)...\left(q^2+1-i\right)...\left(q^2+1-w\right)} =$$

$$= \prod_{i=0}^{w} \frac{q^2+q+2-2i}{q^2+1-i} = \prod_{i=0}^{w}\left(1-\frac{i-1-q}{q^2+1-i}\right) < \prod_{i=0}^{w}\left(1-\frac{i-1-q}{q^2+1}\right).$$

From (5), using the inequality $1 - x \leq e^{-x}$ for $x \neq 0$, we obtain that

$$\pi < e^{-\sum_{i=0}^{w}(i-1-q)/(q^2+1)} = e^{-(w^2-(2q+1)w-2q-2)/2(q^2+1)}.$$

Under the condition

$$(6) \qquad w > \frac{4q^2+2q+2}{2q-1} = 2q+2+\frac{4}{2q-1},$$

it holds that

$$-\frac{w^2-(2q+1)w-2q-2}{2(q^2+1)} < -\frac{(w-2q)^2}{2(q+1)^2},$$

whence

$$\pi < e^{-(w^2-(2q+1)w-2q-2)/2(q^2+1)} < e^{-(w-2q)^2/2(q+1)^2}.$$

The set $\mathcal{K}_{w+1}$ is not ACQ-subset if at least one point of PG(3, q) \ $\mathcal{Q}$ is not covered by it. As |PG(3, q) \ $\mathcal{Q}$| = $q^3 + q$, we have

$$\text{Prob}(\mathcal{K}_{w+1} \text{ is not ASQ-subset}) \le \sum_{A\in PG(3,q)\setminus Q} \text{Prob}\big(A \text{ not covered}\big) \le$$

$$\le (q^3 + q)\pi < (q + 1)^3 e^{-(w-2q)^2/2(q+1)^2}.$$

The probability that all the points of PG(3, q) \ $\mathcal{Q}$ are covered by $\mathcal{K}_{w+1}$ is

$$\text{Prob}(\mathcal{K}_{w+1} \text{ is ACQ-subset}) > 1 - (q + 1)^3 e^{-(w-2q)^2/2(q+1)^2}.$$

This probability is larger than 0 if one takes $w - 2q = \left\lceil (q+1)\sqrt{6\ln(q+1)}\,\right\rceil$, where the condition (6) holds. This shows that there exists an ACQ-subset $\mathcal{K}_{w+1}$ with size

$$w + 1 \le (q+1)\sqrt{6\ln(q+1)} + 2q + 2.$$

Theorem 1(i) is proved.

3. An upper bound on the smallest size of an almost complete cap in PG(N, q)

Assume that in PG(N, q), $N \ge 2$, a cap is constructed by a step-by-step greedy algorithm (*Algorithm*, for short) which in every step adds to the cap a point providing the maximal possible (for the given step) number of new covered points. Such approach is considered in [1, 2, 6].

After the w-th step of Algorithm, a w-cap $\mathcal{K}_w$ is obtained that does not cover exactly U_w points.

Denote by $\mathcal{U}(\mathcal{K})$ the set of points of PG(N, q) that are not covered by a cap $\mathcal{K}$. By above, $\#\mathcal{U}(\mathcal{K}_w) = U_w$. Let the cap $\mathcal{K}_w$ consist of w points $A_1, A_2, \dots, A_w$. Let $A_{w+1} \in \mathcal{U}(\mathcal{K}_w)$ be the point that will be included into the cap in the (w + 1)-st step.

A point A_{w+1} defines a bundle $\mathcal{B}(A_{w+1})$ of w unisecants to $\mathcal{K}_w$ which are denoted as $\overline{A_1A_{w+1}}, \overline{A_2A_{w+1}}, \dots, \overline{A_wA_{w+1}}$, where $\overline{A_iA_{w+1}}$, is the unisecant connecting A_{w+1} with the cap point A_i. Every unisecant contains $q + 1$ points. Except for $A_1, \dots, A_w$, all the points on the unisecants in the bundle are candidates to be new covered points in the (w + 1)-st step. We call $\{A_{w+1}\}$ and $\mathcal{B}(A_{w+1}) \setminus (\mathcal{K}_w \cup \{A_{w+1}\})$, respectively, the *head* and the *basic part* of the bundle $\mathcal{B}(A_{w+1})$. For a given cap $\mathcal{K}_w$, in total, there are $\#\mathcal{U}(\mathcal{K}_w) = U_w$ distinct bundles.

Let $\Delta_w(A_{w+1})$ be the number of new covered points in the (w + 1)-st step, i.e.,

(7) $$\Delta_w(A_{w+1}) = \#\mathcal{U}(\mathcal{K}_w) - \#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}).$$

In future, we consider continuous approximations of the discrete functions $\Delta_w(A_{w+1})$, $\#\mathcal{U}(\mathcal{K}_w)$, $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\})$, keeping the same notations.

We take into account that *all points that are not covered* by a cap *lie on unisecants* to the cap.

In total there are $\theta_{N-1,q}$ lines through every point of PG(N, q). Therefore, through every point A_i of $\mathcal{K}_w$, there is a pencil $\mathcal{P}(A_i)$ of $\theta_{N-1,q} - (w - 1)$ unisecants to $\mathcal{K}_w$, where $i = 1, 2, \dots, w$. The total number T_w^Σ of the unisecants to $\mathcal{K}_w$ is

(8) $$T_w^\Sigma = w(\theta_{N-1,q} + 1 - w).$$

Let $\gamma_{w,j}$ be the number of uncovered points on the j-th unisecant $\mathcal{T}_j$, $j = 1, 2, \dots, T_w^\Sigma$.

Every uncovered point lies on exactly w unisecants; due to this *multiplicity*, on all unisecants there are in total Γ_w^Σ uncovered points, where

(9) $$\Gamma_w^\Sigma = \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = wU_w.$$

By (8), (9), the average number γ_w^{aver} of uncovered points on a unisecant is

(10) $$\gamma_w^{\text{aver}} = \frac{\Gamma_w^\Sigma}{T_w^\Sigma} = \frac{U_w}{\theta_{N-1,q} + 1 - w}.$$

A *unisecant* $\mathcal{T}_j$ *belongs to* $\gamma_{w,j}$ *distinct bundles*, as every uncovered point on $\mathcal{T}_j$ may be the head of a bundle. Moreover, $\mathcal{T}_j$ provides $\gamma_{w,j}(\gamma_{w,j} - 1)$ uncovered points to the basic parts of all these bundles. The noted points are counted with *multiplicity*.

Taking into account the multiplicity, in all U_w the bundles there are

(11) $$\sum_{A_{w+1}} \Delta_w(A_{w+1}) = U_w + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}(\gamma_{w,j} - 1),$$

uncovered points, where U_w is the total numbers of all the heads. By (9), (11),

$$\sum_{A_{w+1}} \Delta_w(A_{w+1}) = U_w + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2 - \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = U_w(1-w) + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2.$$

For a cap $\mathcal{K}_w$, we denote by $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ the average value of $\Delta_w(A_{w+1})$ by all $\#\mathcal{U}(\mathcal{K}_w)$ uncovered points A_{w+1}, i.e.,

(12) $$\Delta_w^{\text{aver}}(\mathcal{K}_w) = \frac{\sum_{A_{w+1}} \Delta_w(A_{w+1})}{\#\mathcal{U}(\mathcal{K}_w)} = \frac{\sum_{A_{w+1}} \Delta_w(A_{w+1})}{U_w} = \frac{\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2}{U_w} - w + 1 \geq 1,$$

where the inequality is obvious by sense; also note that

(13) $$\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2 \geq \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = wU_w.$$

We denote a lower estimate of $\Delta_w^{\text{aver}}(\mathcal{K}_w)$, see Lemma 1 below, as follows:

(14) $$\Delta_w^{\text{rigor}}(\mathcal{K}_w) := \max\left\{1, \frac{wU_w}{\theta_{N-1,q} + 1 - w} - w + 1\right\} =$$

$$= \begin{cases} \dfrac{wU_w}{\theta_{N-1,q} + 1 - w} - w + 1 & \text{if} \quad U_w \geq \theta_{N-1,q} + 1 - w, \\ 1 & \text{if} \quad U_w < \theta_{N-1,q} + 1 - w. \end{cases}$$

Lemma 1. For a w-cap $\mathcal{K}_w$, the following holds:

• This inequality always holds

(15) $$\Delta_w^{\text{aver}}(\mathcal{K}_w) \geq \Delta_w^{\text{rigor}}(\mathcal{K}_w).$$

•• In (15), we have the equality

(16) $$\Delta_w^{\text{aver}}(\mathcal{K}_w) = \Delta_w^{\text{rigor}}(\mathcal{K}_w) = \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1,$$

if and only if every unisecant contains the same number $\dfrac{U_w}{\theta_{N-1,q}+1-w}$ of uncovered points where $\dfrac{U_w}{\theta_{N-1,q}+1-w}$ is integer.

••• In (15), the equality

(17) $$\Delta_w^{\text{aver}}(\mathcal{K}_w) = \Delta_w^{\text{rigor}}(\mathcal{K}_w) = 1,$$

holds if and only if each unisecant contains at most one uncovered point.

P r o o f . By Cauchy-Schwarz-Bunyakovsky inequality, it holds that

$$\left(\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}\right)^2 \leq T_w^\Sigma \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2,$$

where equality holds if and only if all $\gamma_{w,j}$ coincide. In this case $\gamma_{w,j} = \dfrac{U_w}{\theta_{N-1,q}+1-w}$ for all j and, moreover, the ratio $\dfrac{U_w}{\theta_{N-1,q}+1-w}$ is integer. Now, by (8), (9), we have

$$\frac{wU_w}{\theta_{N-1,q}+1-w} \leq \frac{\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2}{U_w},$$

that together with (9), (12), (13), (14) gives (15)-(17). □

Remark 1. One can treat the estimates (15), (16) as follows. A bundle contains w unisecants having a common point, its head. Therefore the average number of uncovered points in a bundle is $w\gamma_w^{\text{aver}} - (w-1)$ where γ_w^{aver} is defined in (10) and the term $w-1$ takes into account the common point.

By (7) and Lemma 1,

$$U_{w+1} \leq U_w\left(1 - \frac{w}{\theta_{N-1,q}+1-w}\right) + w - 1 < U_w\left(1 - \frac{w}{\theta_{N-1,q}}\right) + w$$

whence

(18) $$U_{w+1} - \theta_{N-1,q} < U_w\left(1 - \frac{w}{\theta_{N-1,q}}\right) + w - \theta_{N-1,q} =$$

$$= U_w\left(\frac{\theta_{N-1,q} - w}{\theta_{N-1,q}}\right) - (\theta_{N-1,q} - w) = \left(1 - \frac{w}{\theta_{N-1,q}}\right)(U_w - \theta_{N-1,q}).$$

By (18)

(19) $$U_2-\theta_{N-1,q}<\left(1-\frac{1}{\theta_{N-1,q}}\right)\left(U_1-\theta_{N-1,q}\right);$$

$$U_3-\theta_{N-1,q}<\left(1-\frac{2}{\theta_{N-1,q}}\right)\left(U_2-\theta_{N-1,q}\right)=$$

$$=\left(1-\frac{2}{\theta_{N-1,q}}\right)\left(1-\frac{1}{\theta_{N-1,q}}\right)\left(U_1-\theta_{N-1,q}\right);$$

$$\dots$$

$$U_{w+1}-\theta_{N-1,q}<\left(1-\frac{w}{\theta_{N-1,q}}\right)\dots\left(1-\frac{2}{\theta_{N-1,q}}\right)\left(1-\frac{1}{\theta_{N-1,q}}\right)\left(U_1-\theta_{N-1,q}\right)=$$

$$=\left(U_1-\theta_{N-1,q}\right)f_q(w),$$

where

$$f_q(w)=\prod_{i=1}^{w}\left(1-\frac{i}{\theta_{N-1,q}}\right).$$

Remark 2. The function $f_q(w)$ and its approximations, including (21), appear in distinct tasks of Probability Theory, e.g., in the Birthday problem (or the Birthday paradox) [5, 13]. Actually, let the year contain $\theta_{N-1,q}$ days and let all birthdays occur with the same probability. Then $P^{\neq}_{\theta_{N-1,q}}(w+1)=f_q(w)$, where $P^{\neq}_{\theta_{N-1,q}}(w+1)$ is the probability that no two persons from $w+1$ random persons have the same birthday. Moreover, if birthdays occur with different probabilities we have $P^{\neq}_{\theta_{N-1,q}}(w+1)<f_q(w)$ [5].

By (19), taking into account that $U_1=\theta_{N,q}-1<\theta_{N,q}=\theta_{N-1,q}+q^N$, we have

(20) $$U_{w+1}<q^N f_q(w)+\theta_{N-1,q}.$$

Using the inequality $1-x\le e^{-x}$ for $x\ne 0$, we obtain

(21) $$f_q(w)<\prod_{i=1}^{w}e^{-i/\theta_{N-1,q}}=e^{-(w^2+w)/2\theta_{N-1,q}}<e^{-w^2/2\theta_{N-1,q}}.$$

Let

(22) $$w=\left\lceil\sqrt{2\theta_{N-1,q}\ln q^n}\right\rceil=\left\lceil\sqrt{2N\theta_{N-1,q}\ln q}\right\rceil\sim q^{\frac{N-1}{2}}\sqrt{2N\ln q}.$$

Then, by (20)-(22),

$$w^2=2\theta_{N-1,q}\ln q^N;$$

$$e^{-w^2/2\theta_{N-1,q}}=\frac{1}{q^N};$$

$$U_{w+1}<\theta_{N-1,q}+1;$$

$$U_{w+1}\le\theta_{N-1,q}.$$

So, the number of points of PG(N, q) not covered by the cap $\mathcal{K}_{w+1}$ is at most $\theta_{N-1,q}$.

We have proved Theorem 1(ii).

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